THE LAPLACE-STIELTJES TRANSFORM OF VECTOR-VALUED FUNCTIONS

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1. Let us consider a Banach space X, X' being its strong dual space, and I^+ the positive half-line, $t \geq 0$.

A vector-valued function $\boldsymbol{\alpha}(t)$, $t \in I^+$, $\boldsymbol{\alpha}(t) \in X$ is called with strongly bounded variation on every compact interval $[a, b]$ (see [3]) if there exists a number M so that

$$\text{l.u.b.} \left[\sum_1^n || \boldsymbol{\alpha}(t_{i+1}) - \boldsymbol{\alpha}(t_i) || \right] \leq M$$

for all the partitions $a = t_1 < t_2 \cdots < t_n = b, 0 \leq a < b < \infty$.

It is called with strongly bounded variation in the sense of Gelfand on every compact interval $[a, b]$, if the set of elements

$$\sum_1^n \epsilon_i [\boldsymbol{\alpha}(t_{i+1}) - \boldsymbol{\alpha}(t_i)]$$

where $n = 1, 2, \cdots, \epsilon_i = \pm 1, a = t_1 < t_2 \cdots < t_n = b, 0 \leq a < b < \infty$ is conditionally compact in X.

In the first case we say that $\boldsymbol{\alpha}(t) \in V_r^{[a,b]}$, in the second, that $\boldsymbol{\alpha}(t) \in V_G^{[a,b]}$.

The theory of the Laplace-Stieltjes transform

$$\mathbf{f}(s) = \lim_{R \to \infty} \int_0^R e^{-st} \, d\boldsymbol{\alpha}(t)$$

when $\boldsymbol{\alpha}(t) \in V_r^{[0,R]}$, for every $R > 0$, is due to E. Hille [3].

In this note we shall give a brief account of the possibility of a theory of the Laplace-Stieltjes transform for the functions in $V_G^{[0,R]}$, $R > 0$. Finally, we shall prove that there exist Banach spaces in which $V_r^{[a,b]} \neq V_r^{[a,b]} \cap V_G^{[a,b]}$, and also that there exist Banach spaces in which $V_G^{[a,b]} \neq V_G^{[a,b]} \cap V_r^{[a,b]}$. That is, Hille's theory of the Laplace-Stieltjes integral for vector-valued functions does not imply our theory and vice-versa.

2. Let $\boldsymbol{\alpha}(t) \in V_G^{[0,R]}$ for every $R > 0$. It is known (see [2]) that if $\boldsymbol{\alpha}(t) \in V_G^{[0,R]}$, for every $R > 0$, then $\boldsymbol{\alpha}(t)$ has at most a countable set of discontinuities all of the first kind. Then, we can normalize $\boldsymbol{\alpha}(t)$:

$$\boldsymbol{\alpha}(t) = \tfrac{1}{2}[\boldsymbol{\alpha}(t+) + \boldsymbol{\alpha}(t-)], \qquad \boldsymbol{\alpha}(0) = \theta.$$

One of Dunford's theorems (see [3] or [1]) shows that for every $R > 0$ and complex s, the Stieltjes integral

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