ALL LINEAR OPERATORS LEAVING THE UNITARY GROUP INVARIANT

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Let M_n denote the linear space of all n-square matrices over the complex numbers. Let L_n be the algebra of all linear transformations on M_n to M_n and let O_n be the unitary group in M_n. We denote by Ω_n the multiplicative semi-group in L_n having the property that $T \in \Omega_n$ if and only if $T(O_n) \subseteq O_n$; that is, Ω_n is the set of linear transformations on M_n to M_n which preserve the unitary property. The purpose of this paper is to discuss the structure of Ω_n. Let Q denote the two-element subgroup of Ω_n consisting of the identity and the transformation σ mapping every A into A' where A' is the transpose of A.

The result of this paper is contained in the following

THEOREM. *Ω_n is a group. $O_n \cdot X\tilde{O}_n$ is a normal subgroup of Ω_n and*

$$\Omega_n/0_n \cdot X\tilde{O}_n = Q. \tag{1}$$

The notation we use here is as follows. By $U \cdot XV$ we mean the *direct product* of U and V in M_n; $O_n \cdot X\tilde{O}_n$ is the direct product of the group O_n with its anti-isomorphic image $\tilde{O}_n$. If $A_j \in M_n$ for $j = 1, \cdots, m$ then $\sum_{j=1}^{m} \dotplus A_j$ is the *direct sum* of the A_j. $V^{(n)}$ will be the unitary n-space of complex n-tuples with inner product $(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i$. A^* denotes the complex conjugate transpose of A. If $T \in L_n$, we will write

$$T = (T_{ij})$$

to mean that the n^2-square matrix T is partitioned into n^2 n-square matrices T_{ij}, $i, j = 1, \cdots, n$. If $A \in M_n$ has real eigenvalues, we denote these as

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

The j-th column vector of $A \in M_n$ will be systematically denoted by $v_j(A)$. The n-tuple of numbers with 1 in position i and 0 elsewhere is ϵ_i. It is clear that we may regard the elements of M_n as n^2-tuples as follows: let r be an integer in $1, \cdots, n^2$ and write $r = q_r n + j_r$ where $0 \leq j_r < n$; if $A \in M_n$ then as an element in $V^{(n^2)}$ let its r-th component be $(v_{q_r}(A), \epsilon_{j_r})$.

It is clear that (1) is the same as saying $T \in \Omega_n$ *if and only if*

$$T(A) = UAV \tag{2}$$

or

$$T(A) = UA'V \tag{3}$$

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