ON THE COINCIDENCE OF GEÖCZE AND LEBESGUE AREAS

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Introduction. We shall consider here continuous mappings (T, A) from an admissible set A of the Euclidean plane E_2 into E_3 (or E_n), and the Lebesgue and Geöcze areas $L(T, A)$ and $V(T, A)$ of (T, A). The identity $L(T, A) = V(T, A)$ for all mappings (T, A) lies particularly deep in surface area theory and is a typical instance of identification of an "upper" area as Lebesgue area, and of a "lower" area as Geöcze area. Actually the proof needs to be given only for A a closed finitely connected Jordan region since the extension to any admissible set A can be made by a standard limiting process (see §5).

The proof of $L(T, A) = V(T, A)$ consists of two steps. First it is proved that $L < \infty$ if and only if $V < \infty$, and this is a consequence of the basic inequalities $V(T_r, A) \leq V(T, A) \leq L(T, A) \leq V(T_1, A) + V(T_2, A) + V(T_3, A)$ involving the Geöcze areas of the projections (T_r, A) of (T, A) on the coordinate planes [1, §18.10]. The second step consists in proving that, if $L < \infty$, then $V = L$. A direct proof of this has been given recently by L. Cesari [1, §24.1].

Another and earlier proof of the latter step for the case A being a closed 2-cell is the one of T. Radó [9]. T. Radó's proof is based on the decomposition of (T, A) into its cyclic elements (mappings), the related process of retraction, the cyclic additivity theorems for both Geöcze and Lebesgue areas, and the Morrey representations theorem for "nondegenerate" surfaces of the type of the 2-cell or of the 2-sphere. The proof given by Radó was apparently restricted to the case of A being a closed 2-cell or 2-sphere by the obvious difficulties of defining in more general situations the concept of retraction and the consequent decomposition of a mapping into "elementary" parts.

Recently L. Cesari [2, 3], by a new process of retraction somewhat more general than usual, defined a concept of fine-cyclic element (mapping) for any mapping (T, A) from a closed finitely connected Jordan region. By the use of plane topology, L. Cesari proved a number of properties of fine-cyclic elements, a characterization of fine-cyclic elements by means of properties of separation, and a fine-cyclic additivity theorem for $L(T, A)$. These results extend known properties of the usual cyclic elements.

Successively Ch. J. Neugebauer [6, 7, 8], in the line of analytic topology, discussed corresponding concepts of retractions and of fine-cyclic elements (sets) for any Peano space of finite degree of multicoherence. (The corresponding concepts for mappings T from the same space follow, as usual, by the consideration of the middle space of T.) This discussion showed that fine-cyclic elements of a Peano space of finite degree of multicoherence (and of mappings from them)

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