INTEGRABILITY OF FOURIER TRANSFORMS FOR UNIMODULAR LIE GROUPS

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0. **Introduction.** Let G be a unimodular Lie group. The theory of integration of operators with respect to a gage, formulated by Segal [6], makes it possible to introduce a simple dual object to G in which integration is possible. We shall write $a \to L_a$ for the left regular representation of G on $L_2(G)$; that is, if a is in G and f is in $L_2(G)$, $(L_a f)(x) = f(a^{-1}x)$. If f is in $L_1(G)$, we write L_f for the operator of left convolution by f, acting on $L_2(G)$. The left ring $\mathfrak{L}$ of G is the W^*-algebra generated by the set of all L_x with x in G or by the set of all L_f with f in $L_1(G)$. The dual gage space $\Gamma = (L_2(G), \mathfrak{L}, m)$ consists of the Hilbert space $L_2(G)$, the W^*-algebra $\mathfrak{L}$, and the dual gage m defined by Segal in [7]. In [8], it is shown that m is the unique gage on $\mathfrak{L}$ such that L_f is integrable when f is a continuous positive definite function in $L_1(G)$, with $m(L_f)$ being $f(e)$, where e is the identity of G. We shall refer to $L_f = \int f(x)L_x\, dx$ as the (abstract) Fourier transform of f. If G is Abelian, L_f is unitarily equivalent to the operator of multiplication by the ordinary Fourier transform of f in a way which preserves the relevant integration theories. It is shown in [8] that a number of connections between a function and its Fourier transform carry over from the Abelian case to the arbitrary unimodular case.

This paper is concerned with functions having integrable Fourier transforms. Of course, it follows from the last paragraph that a linear combination of continuous positive definite functions in $L_1(G)$ has an integrable Fourier transform, but this criterion is hard to apply. If G is a vector group, then any function f with compact support which has sufficiently many derivatives has an integrable Fourier transform. For the Fourier transform of $(1 - \Delta)^k f$ is, except for constant factors, $(1 + r^2)^k F$ where Δ is the Laplacian, F is the ordinary Fourier transform of f, and r is distance from the origin in the dual space to G. Since $(1 + r^2)^k F$ is square-integrable, the fact that $(1 + r^2)^{-k}$ is square-integrable for suitable k implies that F is integrable. In fact, $(1 + r^2)^{-k}$ is integrable for suitable k. The function $(1 + r^2)^{-k}$ is the Fourier transform of an "elementary solution" of the differential operator $(1 - \Delta)^k$. Using a theorem of John [3] on the existence of local elementary solutions and a method of Gårding [1] for extending local elementary solutions to global ones, we are able to show that a similar situation holds when G is a unimodular Lie group. A certain element of the universal enveloping algebra of G has an interpretation as a self-adjoint operator whose inverse is L_φ for a continuous, positive-definite, positive, integrable function φ on G. It follows that to establish the integrability of the Fourier

Received August 23, 1957. This work was supported in part by the Office of Naval Research, Contract N6ori-02053.