INVARIANT SUBSPACES OF A NORMAL OPERATOR

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Introduction. In the work that follows, we consider a bounded normal operator on a complex Hilbert space. In general, the terminology and notation agrees with that found in [3]. Our principal interest is the discovery of conditions on the spectrum of a normal operator which imply that the operator has invariant non-reducing subspaces. A subspace K of a Hilbert space H is *invariant* with respect to the operator A if for $x \varepsilon K$, $Ax \varepsilon K$. A subspace which is invariant with respect to A and its adjoint A^* is *reducing*. We were led to this study by the work of Halmos [5, 6], Wermer [14, 15], and Bram [1].

In 1950 Halmos published a paper which initiated the study of what he called subnormal operators. An operator B is *subnormal* on the Hilbert space K provided there is a normal operator A on a Hilbert space H with $K \subset H$ and A restricted to K (written $A \mid K$) is B. Notice that the operator B is assumed given in this definition. We consider as given the operator A. One of our purposes is to find criteria for deciding when A is the extension of a subnormal operator. Particularly, we ask the question, "When is A the extension of a subnormal operator B such that B is not normal?" Since $A \mid K = B$, K is an invariant subspace of H. If K is also reducing, then $A^* \mid K = B$. This would imply that B is normal. Thus our question reduces to "When does A have an invariant non-reducing subspace?" A partial answer to this question is provided by Theorems 2 and 3.

Wermer has studied the problem from a different standpoint [14]. He has considered a normal operator A and what he calls property (P). A normal operator A has *property* (P) if every invariant subspace of A reduces A. The proofs of his Theorems 2 and 3 suggested the use of Borel series for the investigation of our question, at least in the case where the spectrum of A is purely point spectrum. The series and expansions of functions using the method of Runge [13; 10] turned out to be fruitful in the study of the spectrum of a normal operator. These methods led to one of our principal theorems, Theorem 5.

Suppose we know that A on H is a normal operator which does not have property (P). Let K be an invariant non-reducing subspace for A. If the spectrum of A has a hole D in it, Bram has shown that either no point of D belongs to the spectrum of $A \mid K$ or D is contained in the spectrum of $A \mid K$ [1; 80]. Some of the results on Borel series (published by Denjoy [2]) indicated that we might be able to use Borel Series to determine some condition on the spectrum of A in order to insure the existence of a subspace K of H such that

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