ON THE MINIMUM OF THE PERMANENT OF A DOUBLY STOCHASTIC MATRIX

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1. **Introduction.** A real $n \times n$ matrix S is called *doubly stochastic* (d.s.) if

$$s_{ij} \geq 0 \tag{1.1}$$

$$\sum_{j=1}^{n} s_{ij} = 1 \qquad i = 1, \cdots, n \tag{1.2}$$

$$\sum_{i=1}^{n} s_{ij} = 1 \qquad j = 1, \cdots, n. \tag{1.3}$$

The problem we investigate here is the determination of the extreme values of the permanent of S for S d.s. More specifically, let K_n be the (convex) set of all $n \times n$ d.s. matrices. K_n^0 will denote the relative interior of K_n in the Euclidean topology. The permanent of X is the function

$$\text{per}\,(X) = \sum_{(i_1, \cdots, i_n)} x_{1i_1} x_{2i_2} \cdots x_{ni_n},$$

where $(i_1, \cdots, i_n)$ runs over all permutations of $(1, \cdots, n)$. We sometimes will write

$$\text{per}\,(X) = \sum_{\sigma} \prod_{i=1}^{n} x_{i\sigma(i)}$$

where the summation extends over all permutations σ. We wish to determine those $S \in K_n$ for which per (S) is a maximum or a minimum. The determination of the minimum is a problem set by B. L. van der Waerden [3] and stated later by König [2]. Our main result is Theorem 3, which regrettably contains only a partial answer to this question.

The authors wish to express their thanks to H. J. Ryser for bringing the problem to their attention, and to the referee for supplying the reference [3] and for correcting certain inaccuracies in the original version of the paper. A first elementary result is contained in

Lemma 1. *For $S \in K_n$,*

$$\text{per}\,(S) \leq 1 \tag{1.4}$$

Received August 22, 1957. Some of the material in this paper was presented by the second author at the Symposium of Combinatorial Designs and Analysis (sponsored by the American Mathematical Society with the aid of the Office of Ordnance Research) held in New York City in April 1958.

This work was done while the first author was a National Research Council-National Bureau of Standards Post-doctoral Research Fellow, 1956–1957. The work of the second author was supported (in part) by the Office of Naval Research.