BASIC SETS OF POLYNOMIALS FOR THE ITERATED LAPLACE AND WAVE EQUATIONS

By E. P. Miles, Jr. and Ernest Williams

Introduction. For each pair of integers, $n \geq 0$ and $m > 0$, and the associated sets of non-negative integers $a_1, a_2, \cdots, a_k$ such that $a_k \leq 2m - 1$ and $\sum_{i=1}^{k} a_i = n$, the set of homogeneous polynomials

$$(1) \qquad P^n_{a_1, a_2, \cdots, a_k} = \sum_{j=0}^{[(n-a_k)/2]} (-1)^j \begin{pmatrix} j + \left[\frac{a_k}{2}\right] \\ \left[\frac{a_k}{2}\right] \end{pmatrix} \nabla^{2j}(x_1^{a_1} x_2^{a_2} \cdots x_{k-1}^{a_{k-1}}) \cdot \frac{x_k^{a_k+2j}}{(a_k + 2j)!}$$

is shown to form a basic set of k variable polyharmonic polynomials of order m (i.e., solutions of $\nabla^{2m} u = 0$). Deletion of the factor $(-1)^j$ in (1) gives an analogous basic set (1′) for the iterated wave equation

$$\left(\sum_{i=1}^{k-1} \frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial x_k^2}\right)^m \quad u = 0.$$

The paper concludes with the expansion of an arbitrary polynomial in terms of the basic set (1).

Resume of related results ($m = 1$). For $m = 1$ and general k the set (1) is equivalent to the authors' basic set of harmonic polynomials from [4] as expressed in the simpler form suggested by their paper [6]. (In these remarks sets are considered equivalent if, for some correspondence between them, paired elements differ by at most a constant factor.) For $m = 1$ and $k = 3$ the set (1) gives a single formulation of a set given by Protter for which he first [7] gave eight different explicit formulas and later [8] gave a representation requiring four formulas. An alternate derivation of (1) for $m = 1$ and any k was recently given by J. Horváth [2] who was unaware of the earlier results described above.

A brief description of the methods employed by Protter, Horváth and the authors to obtain these harmonic sets follows. Protter obtained the polynomials of his harmonic set as coefficients of $\alpha^a \beta^b \gamma^c$, $a + b + c = n$ in the series expansion of the exponential generating function $e^{\alpha x + \beta y + i\gamma z}$ from which all powers of γ higher than the first had been eliminated by means of the relation $\gamma^2 = \alpha^2 + \beta^2$. Horváth used an analogous approach, obtaining his set as the coordinates of $(x_1 e_1 + x_2 e_2 \cdots + x_k e_k)^n$ with respect to the basis

Received January 29, 1958. The portion of the research done by the first author was supported in part by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract #AF 49 (638) - 81, at the University of Maryland.