THE ADJOINT MARKOFF PROCESS

By Edward Nelson

1. **Introduction.** The theory of Markoff processes is largely concerned with the properties of an order-preserving linear transformation P on a space of functions and its adjoint P^* acting on measures. Since P and P^* act on essentially different types of spaces, the question of self-adjointness or the use of Hilbert space methods would appear to be irrelevant and devoid of probabilistic interest. However, we will show that when there is an invariant measure—i.e., a positive measure m (either finite or infinite) such that $P^*m = m$—the definitions of P and P^* may be extended in a natural way so that they act on the same spaces. The probabilistic interpretation of P^* acting on functions or P acting on measures is that of the original process with the time direction reversed.

In the following section we consider the existence of an invariant measure. The adjoint Markoff process of a Markoff process with an invariant measure is constructed in §3. The question of uniqueness of the invariant measure is closely connected with recurrence properties of the sample functions. In §4 an analytic criterion for recurrence is obtained, and this is applied in §5 to show the uniqueness of the invariant measure of a recurrent process. This proof shows one use of the adjoint process: it enables us to replace a question about measures by a question about functions to which the martingale convergence theorem gives an answer. In §6 the adjoint of a diffusion process is examined in the simplest case, that in which the infinitesimal generator is a smooth partial differential operator on a compact manifold.

2. **The existence of an invariant measure.** Our approach to invariant measures will be by means of invariant linear functionals. To connect the two we need a topology on the underlying space. Throughout this section X will be a locally compact Hausdorff space, B a linear space of bounded continuous functions on X containing the space C of all continuous functions vanishing at infinity, C_0 those functions in C which have compact support, B^+ those functions in B which are positive and not identically 0, $C^+ = C \cap B^+$, and $C_0^+ = C_0 \cap B^+$. We will use $\| f \|$ to mean $\sup_{x \in X} | f(x) |$, and $\| P \| = \sup_{\|f\|=1} \| Pf \|$ for a linear transformation P on B. The measure-theoretic terminology is that of Halmos [5].

Theorem 2.1. *Let P be a linear transformation of B into itself which carries B^+ into itself and is such that $\| P \| \leq 1$, and $P1 = 1$ if X is compact. If for all*

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