ON A THEOREM OF TSCHEBOTAREFF

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Introduction. Let $L_1, L_2, \cdots, L_n$ be linear forms in the variables $u_1, u_2, \cdots, u_n$ with real coefficients a_{ij} such that $|\det(a_{ij})| = 1$. For a given set of real numbers $c_1, c_2, \cdots, c_n$ put

$$M = \inf \prod_{i=1}^{n} | L_i + c_i |$$

extended over all rational integral values of the variables $u_1, u_2, \cdots, u_n$. A conjecture of Minkowski's asserts that $M \leq 2^{-n}$. This has been proved for $n = 2, 3, 4$, but the general case remains unsolved. In this direction the most important result obtained so far is due to Tschebotareff [3] who proved that $M < 2^{-\frac{1}{2}n}$. Improvements have been obtained by Mordell [2]: $M < (2^{\frac{1}{2}n} + (2 - 2^{\frac{1}{2}})^n)^{-1}$, and Davenport [1]: $M < (\gamma_n 2^{\frac{1}{2}n})^{-1}$ where $\gamma_n > 1$ and $\lim_{n \to \infty} \gamma_n = 2e - 1$.

This paper is divided into two sections. The first is occupied with showing that Minkowski's conjecture is equivalent to a conjecture on the precise value of the critical determinant of a certain n-dimensional region, a result implicit in the work of others, notably Tschebotareff [3], see also Davenport [1]. For the sake of completeness some immediate corollaries of this result are given though in point of fact they are well known. The second section is concerned with applying this theorem to show that

$$M \leq 2^{-\frac{1}{2}n}(2 - (2 - \sqrt{2})^n)^{-1}.$$

This bound is smaller than that of Mordell's for every n, and to my knowledge represents the best known explicit bound for moderate values of $n \geq 5$.

1. **An equivalent conjecture.** Let $(x_1, x_2, \ldots, x_n)$ be Cartesian coordinates of a point X in R_n. Denote by K_1 the set of points X for which

$$\prod_{i=1}^{n} | x_i + 1 | \leq 1$$

and by K_2 the set of points X for which

$$\prod_{i=1}^{n} | x_i - 1 | \leq 1. \tag{1}$$

Put $K_* = K_1 \cup K_2$.

Theorem 1. $M \leq \Delta(K)^{-1}$.

Proof. There is no loss of generality in assuming that $M > 0$. By the definition of M we may associate with each positive integer r a non-negative real

Received November 26, 1957. This work was completed at the Victoria University of Manchester.