# NOTE ON DIFFERENTIATING MARKOFF TRANSITION FUNCTIONS WITH STABLE TERMINAL STATES

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By a Markoff function we shall mean an element of the matrix $\{p_{ij}(t)\}$ where $p_{ij}(t)$ is a single real-valued function on $0 \leq t < \infty$ and

I $$0 \leq p_{ij}(t) \leq 1$$

II $$\sum_j p_{ij}(t) = 1$$

III $$p_{ij}(t_1 + t_2) = \sum_k p_{ik}(t_1)p_{kj}(t_2)$$

IV $$\lim_{t \to 0^+} p_{ij}(t) = \delta_{ij} = p_{ij}(0).$$

We recall several pertinent facts concerning such a matrix. The right-hand derivative at 0 of $p_{ij}(t)$ exists for all $j$ and is finite providing $j \neq i$ (see [4] and [6]). We denote the derivative $Dp_{ij}(0)$ by $q_{ij}$ , $j \neq i$, $Dp_{ii}(0) = -q_i$ ; if $q_i < \infty$, the state $i$ is said to be stable. In [1] we show that $p_{ij}(t)$ possesses a continuous derivative for all $j$ and $t$ (further $| Dp_{ii}(t) | \leq q_i$) satisfying the differential equation

$$(1) \qquad Dp_{ij}(t_1 + t_2) = \sum_k Dp_{ik}(t_1)p_{kj}(t_2), \qquad t_1 > 0.$$

In this note we establish the analogous result that *if $j$ is a stable state, then $p_{ij}(t)$ has a continuous derivative for all $i$ and $t$ and the differential equation*

$$(2) \qquad Dp_{ij}(t_1 + t_2) = \sum_k p_{ik}(t_1)Dp_{kj}(t_2)$$

holds for all $t_1 \geq 0$ and $t_2 > 0$. This result appeared in our technical report [2]. However, the proof presented here has been shortened. In particular, we make use of the following lemma due to Chung [3; 208]. (This result is contained in a proof and not explicitly stated.)

Lemma. *If $u_{kj}(t)$ are non negative and satisfy $u_{ij}(t + s) \geq \sum_k p_{ik}(s)u_{kj}(t)$ where equality holds for almost all $t$ for any $s$, then equality always holds.*

The differential equation (2) has been obtained by Chung [3] under the hypothesis that all states are stable. The probabilistic significance of stability is discussed in [3], [4], and [7].

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