THE PROBLEM OF MILLOUX FOR FUNCTIONS ANALYTIC IN AN OPEN ANNULUS

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1. **Introduction.** The problem of Milloux for an annulus in the complex plane may be stated as follows: Let $A = [0 < s < |z| < 1]$, $C_\rho = [|z| = \rho]$, $C = C_s \cup C_1$, and let $\mathcal{E}$ be a point set in A, closed relative to A, and such that $C_\rho \cap \mathcal{E} \neq \phi$ for all ρ satisfying: $s < \rho < 1$. With $0 < m < 1$, let $\mathfrak{F}_m$ denote the family of functions g which are defined, analytic, and of modulus less than one in A, and each one of which has an associated $\mathcal{E}$-set such that $z \in \mathcal{E}$ implies $|g(z)| \leq m$. Finally, let $M(g; r) = \max_{|z|=r} |g(z)|$. The problem of Milloux is to determine, for fixed r such that $s < r < 1$,

$$\text{l.u.b.}_{g \in \mathfrak{F}_m} M(g; r) \tag{1.1}$$

and the associated extremal functions. Since $\mathfrak{F}_m$ is a normal family, an extremal function satisfying (1.1) exists, and it is a member of $\mathfrak{F}_m$.

Historical warrant for the name assigned to the problem is contained in the pioneer work of Milloux [5] on a cognate problem. The above statement of the problem is the analogue for the annulus of the formulation of the Milloux problem for the open unit disk which is due to Heins [2].

The present paper establishes the following conclusions: 1) if an extremal function f for (1.1) is normalized by the requirement that $f(r) = M(f; r)$, then f is unique, is independent of r, is real-valued for z real, and has $[-1 < z < -s]$ for an associated $\mathcal{E}$-set; 2) if ϕ is the similarly normalized, unique extremal for the corresponding problem for $[|w| < 1]$, and if ψ is a certain specific (1, 2) analytic map of A onto $[|w| < 1]$, then $f = \phi \circ \psi$; 3) f may be represented as the quotient of θ-functions or of S-functions of Rausenberger.

2. **Extremal function for the Milloux problem for** $[|w| < 1]$. The following results obtained by Heins [2] will be of critical importance in the solution of the Milloux problem for the annulus.

Let E be a set of points in $[|w| < 1]$, closed relative to this domain, which satisfies: $C_\rho \cap E \neq \phi$ for all ρ such that $0 \leq \rho < 1$. With $0 < m < 1$, let F_m be the family of functions g which are defined, analytic, and of modulus less than one on $[|w| < 1]$, and each one of which has an E-set associated with it by the condition that $w \in E$ implies $|g(w)| \leq m$. Then, for fixed positive r less than one, $\text{l.u.b.}_{g \in F_m} M(g; r)$ is attained by a member of F_m. If $\phi = \phi(w; m, r)$

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