LOCALLY o-CONVEX SPACES

By Joseph Kist

1. **Introduction.** A non-empty subset S of a partially ordered vector space is called *o-convex* if it is convex, and if $x \in S$ whenever there exist elements s and s' in S such that $s \leq x \leq s'$. A *locally o-convex space* is a partially ordered vector space together with a locally convex topology for which there exists a fundamental system of neighborhoods of the origin consisting of o-convex and symmetric sets. The most comprehensive study of such spaces has been made by Namioka [13]. The main objective of this paper is to characterize those locally o-convex spaces E which have the property that every homomorphism of E with range in any locally o-convex space is continuous. This is done in §7.

Definitions and preliminary results are collected in §2. If E is any partially ordered locally convex space, then the collection of all those neighborhoods of the origin which are o-convex and symmetric forms a base at the origin for a coarser locally o-convex topology on E. Some relationships between the latter topology and the original topology on E are obtained in §3. In §4, it is shown that every element of the dual of a locally o-convex space can be expressed as the difference of two continuous positive linear functionals, thereby generalizing, in one direction, a result of M. Krein. This result has also been obtained, independently, by Bonsall and Namioka. In preparation for our study of two analogues of bornological spaces (§§6 and 7), o-inductive limits are defined and discussed in §6.

A vector lattice, equipped with a compatible topology (i.e., a locally convex topology for which there exists a fundamental system $\{V\}$ of neighborhoods of the origin such that $x \in V$ and $|y| \leq |x|$ imply $y \in V$), is a member of the class of locally o-convex spaces. In §8, we obtain necessary and sufficient conditions that a compatible topology coincide with the finest compatible topology, thereby solving a problem which was suggested by Professor Casper Goffman during a conversation with the author.

2. **Preliminaries.** Regarding the theory of locally convex spaces, we use the terminology and results of [5] and [6]. The scalar field of all vector spaces considered is the field of real numbers. A non-empty subset Q of a vector space is called a *cone* provided that (i) $Q + Q \subset Q$, and (ii) $tQ \subset Q$ for all non-negative scalars t. A *partially ordered vector space* (POVS), denoted by (E, P), is a vector space E together with a cone P such that (iii) $P \cap (-P) = \{0\}$, where 0 denotes the origin of E; we write $x \leq y$ (or $y \geq x$) if $y - x \in P$, and

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