FUNCTIONAL INVARIANTS OF A LINEAR HOMOGENEOUS INTEGRO-DIFFERENTIAL EQUATION

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In his book, *Projective Differential Geometry of Curves and Ruled Surfaces*, Wilczynski obtained the semi-invariants, invariants and covariants of a system of two differential equations in two dependent variables. More recently Barnett and Reingold [2] treated this problem for n equations in n dependent variables and found special types of invariants. Later Levine [4] and Abian [1] further generalized this problem and gave a method for finding invariants and covariants of general order r. In 1950 Barnett [3] proposed the problem of finding functional invariants of an integro-differential equation of the second order under Fredholm transformation, and gave some simple examples. The purpose of the present paper is to further extend these last results to homogeneous integro-differential equations of order $m \geq 2$.

1. **Statement of the problem.** Consider the linear homogeneous integro-differential equation

$$y^{(m)}(x;\,t) + \int_0^1 a_1(x,\,u;\,t)y^{(m-1)}(u;\,t)\,du + \cdots + \int_0^1 a_m(x,\,u;\,t)y(u;\,t)\,du = 0, \tag{1}$$

where t is a real parameter ranging on $[t_1\,,\,t_2]$ and such that for each t the given kernels $a_i(x,\,u;\,t)$ are continuous on the unit square, and the function $y(x;\,t)$ is continuous on the unit interval. The i-th derivative of $y(x;\,t)$ with respect to t is denoted by $y^{(i)}(x;\,t)$. It is assumed also that the functions depending on t possess as many derivatives with respect to t as needed.

Consider also the Fredholm transformation of $y(x;\,t)$ given by

$$y(x;\,t) = \bar{y}(x;\,t) + \int_0^1 k(x,\,u;\,t)\bar{y}(u;\,t)\,du, \tag{2}$$

where for each t the kernel $k(x,\,u;\,t)$ is continuous on the unit square, and has a non-vanishing Fredholm determinant.

The transformation $y(x;t) \rightarrow \bar{y}(x;t)$ given by (2) transforms (1) into an equation of the same type with new coefficients $\bar{a}_i(x,\,u;\,t)$. If f is a functional involving the $a_i(x,\,u;\,t)$ and their derivatives with respect to t, a functional $\bar{f}$ can be formed by replacing $a_i(x,\,u;\,t)$ and their derivatives by $\bar{a}_i(x,\,u;\,t)$ and the corresponding derivatives of $\bar{a}_i(x,\,u;\,t)$ with respect to t. The problem is then to find functionals f such that $f = \bar{f}$.

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