A CLASS OF SIMPLE GAMES

By J. R. Isbell

Introduction. It is known [3] that every strong simple game has at least as many minimal winning sets as non-dummy players. This paper is a complete determination of the cases of equality; thus it is a sequel to a paper [5] in which I determined all weighted majority games having the same number of minimal winning sets as of non-dummy players. The complete list consists of the infinite family of [5] and the seven-player projective game. (A simple game is *projective* if its players and minimal winning sets satisfy the incidence axioms for points and lines in a projective plane.)

The method of proof involves dividing the games in question into two classes and showing that all the games in one class are of the type studied in [5], and all the games in the other class are projective. The classification is completed by appealing to Richardson's proof [8] that the only strong projective game is the one corresponding to the projective plane of seven points.

§1 recalls the definitions, including Gillies' unpublished definition of a *pseudogame* (from his thesis [2]), and some known results which we need. §2 develops two operations on games called restriction and reduction. Reduction of an n-player game consists of combining some two of its players into one, so as to make an $(n - 1)$-player game, and to reduce (strictly) the number of minimal winning sets. This is not always possible, and in fact our fundamental dichotomy is the distinction between *reducible* and *irreducible* games. Restriction is a more complicated operation which takes us into pseudogames. Combining the results on restriction and on reduction, we arrive at the following lemma:

If G is an irreducible strong simple game of four or more players and the same number of minimal winning sets, then every two minimal winning sets of G have just one player in common. From a theorem of de Bruijn and Erdös [1], and a remark in M. Hall's review of [1], it follows that G is projective; and by Richardson's theorem [8], and a separate treatment of the cases $n = 1, 2, 3$, the classification in the irreducible case is completed. This is §3; §4 disposes of the less interesting reducible case. Also, some of the details on restrictions and pseudogames are gathered in an appendix following §4, so that the most interesting material is all in the first half of the paper. This includes a new proof of the theorem of Gurk and Isbell [3] on the number of minimal winning sets, and a new proof of the theorem of deBruijn and Erdös [1] on the number of lines.

1. Background.

The next four paragraphs are not part of the formal argument, but a partially heuristic sketch of the concepts and of the first half of the proof. Some readers may prefer to skip to the definitions.

Received October 4, 1957. Research supported by the National Science Foundation.