REPRESENTATIONS OF EVEN FUNCTIONS (MOD r), I. ARITHMETICAL IDENTITIES

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1. **Introduction.** Let n and r denote integers with $r > 0$ and let $f(n) = f(n, r)$ be an arithmetical function with values in the complex field. We say that $f(n, r)$ is an *even* (periodic) function of $n(\text{mod } r)$ if $f(n, r) = f((n, r), r)$ for all values of n. The theory of even functions (mod r) was discussed in [9]; in the present paper we propose to develop further the properties of this class of functions. Related aspects of the theory will be developed in a sequel to this paper to be published elsewhere. A significant consequence of the investigation is an elementary and unified method for obtaining identities between arithmetical functions. Before summarizing the content of the paper we recall the main results on even functions (mod r) proved in [9].

The class of even functions (mod r) was characterized in two ways. In the first place, it was shown that $f(n, r)$ is even (mod r) if and only if it possesses a Fourier expansion of the form,

$$f(n, r) = \sum_{d \mid r} \alpha(d, r) c(n, d); \tag{1.1}$$

here $c(n, r)$ denotes the familiar trigonometric sum of Ramanujan,

$$c(n, r) = \sum_{(x, r) = 1} e(nx, r), \qquad (e(n, r) = e^{2\pi i n/r}),$$

the summation extending over a reduced residue system (mod r). Furthermore, the Fourier coefficients $\alpha(d, r)$ are determined by the formula,

$$\alpha(d, r) = \frac{1}{r} \sum_{\delta \mid r} f\left(\frac{r}{\delta}, r\right) c\left(\frac{r}{d}, \delta\right). \tag{1.2}$$

This result, in an equivalent form, was proved by Ramanathan [23, Lemma C]. A second expression for $\alpha(d, r)$, useful in applications of an additive nature, was proved in [9, (8)].

It was also shown in [9] that $f(n, r)$ is even (mod r) if and only if it possesses an arithmetical representation of the form,

$$f(n, r) = \sum_{d \mid (n, r)} g\left(d, \frac{r}{d}\right), \tag{1.3}$$

where $g(a, b)$ is an arithmetical function of two variables and the summation ranges over the common divisors of n and r. Moreover, if $f(n, r)$ is defined by (1.3), then the Fourier coefficients $\alpha(d, r)$ are determined by

$$\alpha(d, r) = \frac{1}{r} \sum_{e \mid r/d} g\left(\frac{r}{e}, e\right) e. \tag{1.4}$$

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