q-BERNOULLI AND EULER NUMBERS OF HIGHER ORDER

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1. Recently Professor L. Carlitz [1], [2] has introduced an interesting analogue of Bernoulli numbers and polynomials. He has also indicated a corresponding Staudt-Clausen theorem and also some interesting congruence properties of the q-Bernoulli numbers. It appears, however, that an extension of these q-Bernoulli numbers to higher order is not so easy, especially because a simple generating function for them is not known. The object of this paper is to define numbers $\beta_m^{(h,k)}$ for h, k integers ≥ 1 which reduce for $h = k$ to the Bernoulli numbers $B_m^{(k)}$ as $q \to 1$. We also define the corresponding numbers $\epsilon_m^{(h,k)}$ which for $h = k$ and $q = 1$ reduce after a suitable modification to $E_m^{(k)}$, the Euler numbers of higher order. We also obtain a sort of Staudt-Clausen theorem when $h > k = 1$, but we withhold these results until some later date.

2. We shall use the following notation:

$$(x)_s = x(x-1) \cdots (x-s+1)$$

$$[x] = \frac{q^x - 1}{q - 1}, \qquad [x]_s = [x][x-1] \cdots [x-s+1]$$

$$[m]! = [m]_m, \qquad \begin{bmatrix} x \\ s \end{bmatrix} = [x]_s/[s]!, \begin{bmatrix} x \\ 0 \end{bmatrix} = 1.$$

Let h, k be positive integers ≥ 1 and let the numbers $\beta_m^{(h,k)}$ be defined by

$$\beta_m^{(h,k)} = \beta_m^{(h-1,k)} + (q-1)\beta_{m+1}^{(h-1,k)} \qquad (m \geq 0) \tag{2.1}$$

$$\beta_m^{(0,k+1)} = \frac{m-k}{[m-k]}\beta_m^{(0,k)} \tag{2.1a}$$

Let us start by observing that for $h = 0$, $k = 1$, the β's are the numbers η and for $h = 1$, $k = 1$, they are the numbers β introduced by Carlitz. Thus we have

$$\beta_m^{(0,1)} = \eta_m, \qquad \beta_m^{(1,1)} = \beta_m .$$

We list here some properties of the numbers η_m and β_m to which we shall have occasion to refer (Carlitz [1]):

$$\begin{cases} \eta_0 = 1, \quad \eta_1 = 0, \quad (q\eta + 1)^m = \eta_m, \quad m > 1 \\ \beta_0 = 1, \quad \beta_1 = -\dfrac{1}{[2]}, \quad \beta_2 = \dfrac{q}{[2][3]}, \\ q(q\beta + 1)^m - \beta_m = \begin{cases} 1 & m = 1 \\ 0 & m > 1 \end{cases} \end{cases} \tag{2.2}$$

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