A BOUNDARY VALUE PROBLEM FOR A SINGULARLY PERTURBED DIFFERENTIAL EQUATION

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1. The boundary value problem ($' = d/dx$)

$$\epsilon y''' = f(x, y, y', y'', \epsilon) \qquad y(0) = a, \qquad y(1) = b, \qquad y'(1) = b_1 \tag{1.1}$$

will be considered under the hypothesis that the degenerate equation (with $\epsilon = 0$)

$$f(x, u, u', u'', 0) = 0 \tag{1.2}$$

has a solution $u = g(x)$ for $0 \leq x \leq x_0$ and $u = h(x)$ for $x_0 \leq x \leq 1$ where $0 < x_0 < 1$ and $g(0) = a$, $h(1) = b$, $h'(1) = b_1$. Further

$$g(x_0) = h(x_0), \qquad g'(x_0) = h'(x_0), \qquad g''(x_0) \neq h''(x_0).$$

The case of a second order equation has been considered [1], but the method used there fails for higher order equations. The method used here can be used for higher order equations.

The case $h''(x_0) > g''(x_0)$ will be considered here. With no restriction it can be assumed that

$$g'(x_0) = h'(x_0) = 0, \qquad -g''(x_0) = h''(x_0) = \mu > 0$$

since the change of variable $\tilde{y} = y - g'(x_0)\,(x - x_0) - \frac{1}{2}[g''(x_0) + h''(x_0)]\,(x - x_0)^2$ will achieve this. Further let $y_0 = g(x_0)$.

Let α, β, and ϵ_0 be positive constants. Let $U(x) = g(x)$ for $0 \leq x \leq x_0$ and let $U(x) = h(x)$ for $x_0 \leq x \leq 1$. Let R be the region of (x, y, w, z, ϵ) space determined by the union of $0 \leq \epsilon \leq \epsilon_0$, $0 \leq \dot{x} < x_0 < x \leq 1$,

$$| y - U(x) | + | w - U'(x) | + | z - U''(x) | \leq \alpha$$

and

$$| x - x_0 | + | y - y_0 | + | w | \leq \alpha, \qquad -\mu - \beta < z < \mu + \beta, \qquad 0 \leq \epsilon \leq \epsilon_0 .$$

In R let $f(x, y, w, z, \epsilon)$ be real-valued and of class C. Let the second order partial derivatives of f with respect to (x, y, w, z) exist and be of class C in R.

Theorem. *Let*

$$f_z(x, g(x), g'(x), g''(x), 0) > 0 \qquad 0 \leq x \leq x_0 \tag{1.3}$$

$$f_z(x, h(x), h'(x), h''(x), 0) < 0 \qquad x_0 \leq x \leq 1 \tag{1.4}$$

$$f(x_0, y_0, 0, z, 0) > 0 \qquad -\mu < z < \mu. \tag{1.5}$$

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