SOME ORTHOGONAL FUNCTIONS IN SEVERAL VARIABLES RELATED TO THETA FUNCTIONS

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1. Let·

$$H_n(x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} x^r, \qquad G_n(x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-n)} x^n, \tag{1.1}$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-r+1})}{(q)_n} \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = 1,$$
$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n), \qquad (q)_0 = 1. \tag{1.2}$$

Szegö [4] showed that

$$\frac{1}{2\pi}\int_0^{2\pi} H_m(-q^{-\frac{1}{2}}e^{i\phi})H_n(-q^{-\frac{1}{2}}e^{-i\phi})f(\phi)\,d\phi = q^{-n}(q)_n\,\delta_{mn}\,, \tag{1.3}$$

where

$$f(\phi) = \sum_{-\infty}^{\infty} q^{\frac{1}{2}n^2}e^{ni\phi} = \sum_{-\infty}^{\infty} q^{\frac{1}{2}n^2}\cos n\phi \qquad (|\,q\,| < 1). \tag{1.4}$$

Wigert [5] showed that

$$\int_0^{\infty} G_m(-q^{m+\frac{1}{2}}x)G_n(-q^{n+\frac{1}{2}}x)p(x)\,dx = q^{-n-\frac{1}{2}}(q)_n\,\delta_{mn}\,, \tag{1.5}$$

where

$$p(x) = k\pi^{-\frac{1}{2}}\exp(-k^2\log^2 x), \qquad 2k^2 = -\frac{1}{\log q}. \tag{1.6}$$

See also Hahn [3].

The object of the present note is to construct certain sequences of functions that are biorthogonal with respect to the N-dimensional theta-function

$$\vartheta(\phi_1\,,\cdots,\phi_N) = \sum_{r_j=-\infty}^{\infty} e^{-\frac{1}{2}Q(r_1\lambda_1,\cdots,r_N\lambda_N)}\cdot e^{(r_1\phi_1+\cdots+r_N\phi_N)i} \tag{1.7}$$

on the one hand, and with respect to the function

$$p(x_1\,,\cdots,x_N) = (\lambda_1\cdots\lambda_N)^{-1}\frac{\Delta^{\frac{1}{2}}}{(2\pi)^{N/2}}(x_1\cdots x_N)^{-1} \cdot\exp\left\{-\frac{1}{2}Q\left(\frac{\log x_1}{\lambda_1}\,,\cdots,\frac{\log x_N}{\lambda_N}\right)\right\}. \tag{1.8}$$

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