THE TOPOLOGY OF COMPACT CONVERGENCE ON CONTINUOUS FUNCTION SPACES

By Seth Warner

Let T be a completely regular space, $\mathcal{C}(T)$ the collection of all continuous real-valued functions on T. $\mathcal{C}(T)$ may be given a variety of structures, and a natural problem is to relate properties of the given structure imposed on $\mathcal{C}(T)$ with topological properties of T. $\mathcal{C}(T)$ may, for example, be regarded as a ring or algebra (without topology) over the reals $\mathbf{R}$, and a lively interest has developed in relating algebraic properties of $\mathcal{C}(T)$ with topological properties of T. We shall be concerned with a different problem: If $\mathcal{C}(T)$ is equipped with the topology of uniform convergence on compact subsets of T, $\mathcal{C}(T)$ becomes a locally convex vector space over $\mathbf{R}$; we shall relate certain properties of this locally convex space with properties of T.

Considerable progress has already been made on this problem, among others by Arens, Nachbin, Shirota, and Myers. We list some of the results obtained thus far:

A *hemicompact* space is one in which there exists a countable family $\mathcal{K}$ of compact subsets such that each compact subset is contained in some member of $\mathcal{K}$ [1]. Arens has proved the following [1, Theorems 7 and 8]:

Theorem A. *$\mathcal{C}(T)$ is metrizable if and only if T is hemicompact.*

Two of the deepest results of the subject are the following two theorems, each proved independently by Nachbin [17] and Shirota [19]:

Theorem B. *$\mathcal{C}(T)$ is bornological if and only if T is a Q-space.*

Theorem C. *$\mathcal{C}(T)$ is barrelled if and only if for every closed non-compact subset S of T, there exists $x \in \mathcal{C}(T)$ which is unbounded on S.*

Additional results have been obtained when T is compact and $\mathcal{C}(T)$, therefore, a Banach space. In 1940, M. and S. Krein announced part of the following theorem [14, Theorem 2] (for a proof of the entire theorem, see Proposition 16 of [9; 19]):

Theorem D. *If T is compact, then $\mathcal{C}(T)$ is separable* (i.e., *has a countable dense subset*) *if and only if T is metrizable.*

As a final example, reflexivity has been characterized by Myers [16, Theorem 4]:

Theorem E. *If T is compact, $\mathcal{C}(T)$ is reflexive if and only if T is finite.*

We shall assume familiarity with Bourbaki's *Topologie Générale* and *Espaces*

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