ALMOST COMPLETELY CONVEX FUNCTIONS

By R. P. Boas, Jr.

An infinitely differentiable function f defined on $[0, 1]$ is called completely convex (Widder) if

$$(1) \qquad f^{(4k)}(x) \geq 0, \qquad f^{(4k+2)}(x) \leq 0, \qquad 0 \leq x \leq 1; \quad k = 0, 1, 2, \cdots .$$

Protter [5] introduced the class of almost completely convex functions, for which (1) is replaced by the less restrictive condition

$$(2) \qquad \begin{cases} f^{(4k)}(x) \geq 0, \\ f^{(4k+2)}(0) + f^{(4k+2)}(1) \leq \pi^2[f^{(4k)}(0) + f^{(4k)}(1)]. \end{cases}$$

He showed that almost completely convex functions share with completely convex functions the property of being restrictions of entire functions of exponential type π. The new class is significantly larger than the old one, including as it does the functions e^{ax} $(0 < a \leq \pi)$ which are far from being completely convex.

Protter's proof is elementary. Here I shall give a less elementary but shorter proof which can be adapted to establish somewhat more general results.

Let f be almost completely convex. By repeated integration by parts, we have (with Protter)

$$(3) \qquad \begin{aligned} M = \int_0^1 f(x) \sin \pi x \, dx &= \pi^{-1}[f(0) + f(1)] - \pi^{-3}[f''(0) + f''(1)] + \cdots \\ &\quad + \pi^{-4k+3}[f^{(4k-4)}(0) + f^{(4k-4)}(1)] \\ &\quad - \pi^{-4k+1}[f^{(4k-2)}(0) + f^{(4k-2)}(1)] + \pi^{-4k} \int_0^1 f^{(4k)}(x) \sin \pi x \, dx. \end{aligned}$$

If (2) holds, it follows that

$$\pi^{-4k} \int_0^1 f^{(4k)}(x) \sin \pi x \, dx \leq M,$$

and since $f^{(4k)}(x) \geq 0$ this implies that if $0 < a < a + b < 1$ we have

$$\int_a^{a+b} | f^{(4k)}(x) | \, dx = \int_a^{a+b} f^{(4k)}(x) \, dx \leq \pi^{4k} M', \qquad k = 0, 1, 2, \cdots ,$$

for some M'. By a theorem of Halperin and Pitt's [3; 617, Theorem 3, $n = 4$, $p = 1$], this implies

Received September 17, 1957. Research supported by the National Science Foundation.