THE HARMONIC SUMMATION OF THE DERIVED FOURIER SERIES

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1. Let $f(\theta)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π and let

$$f(\theta) \sim \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta) = \tfrac{1}{2}a_0 + \sum_1^\infty A_n(\theta). \tag{1.1}$$

Then the differentiated series of (1.1) at $\theta = x$ is

$$\sum_1^\infty n(b_n \cos nx - a_n \sin nx) = \sum_1^\infty nB_n(x). \tag{1.2}$$

Write

$$\psi(t) = f(x + t) - f(x - t), \qquad g(t) = \psi(t)/4 \sin \tfrac{1}{2}t - C, \tag{1.3}$$

where C is a function of x.
Let S_n and t_n be respectively the n-th partial sum and first harmonic mean of the series (1.2) so that

$$S_n = \sum_{r=1}^{n} rB_r(x), \tag{1.4}$$

$$t_n = \sum_{r=1}^{n} S_{n-r}/r \log n. \tag{1.5}$$

Definition. The series (1.2) is said to be summable $(N, 1/n)$ to C, provided that $t_n \to C$ as $n \to \infty$.

Our object in this note is to prove the following

Theorem. *If $g(t)$ is of bounded variation in $(0, \pi)$, and $g(t) \to 0$ as $t \to 0$, then the series (1.2) is summable $(N, 1/n)$ to the value C.*

In the above theorem it is enough to consider the special case in which $C = 0$. To justify this assertion, consider first the case in which

$$f(\theta) = C \sin (\theta - x).$$

Then $B_1(x) = C$, $B_n(x) = 0$ $(n \geq 2)$, so that the series (1.2) converges to C and therefore is summable $(N, 1/n)$ a fortiori.

In the general case write

$$f(\theta) = C \sin (\theta - x) + f_1(\theta).$$

Let $g_1(\theta)$ be formed from $f_1(\theta)$ in the same way as $g(\theta)$ from $f(\theta)$ but with the C corresponding to $f_1(\theta)$ taken as 0. Then

$$g_1(t) = g(t) + C(1 - \cos t/2).$$

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