A CHARACTERIZATION OF THE CLASSICAL GROUPS

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1.1. **Introduction.** By one method of classification there are three types of (complex, connected) classical groups, (a) $GL(n. C)$, (b) $SO(n, C)$, and (c) $Sp(n, C)$. So designated, each type is given as a specific group of matrices. It is perhaps neater (and for us more pertinent) to describe these groups by means of the special linear representation which each type admits. That is, if V is a complex vector space of dimension m, then up to an inner automorphism $GL(m, C)$ is in a natural way isomorphic to the group $GL(V)$ of all non-singular linear transformations on V. If $\mathbf{B}$ is a non-singular bilinear form on V, then the identity component of the subgroup of $GL(V)$ which leaves $\mathbf{B}$ invariant is in a natural way (again up to an inner automorphism) isomorphic to $SO(m, C)$ in case $\mathbf{B}$ is symmetric and to $Sp(n, C)$ in case $m = 2n$ and $\mathbf{B}$ is skew-symmetric.

If G is a classical group, then the isomorphism $\pi\colon G \to GL(V)$ described above defines a representation of G which we shall call a natural representation. It is a simple property that a natural representation is irreducible and that any two such representations are equivalent. (Hence, of course, such a representation defines only one member, but a special member, of the infinite collection of equivalence classes of equivalent irreducible representations of G.)

We shall call a subgroup G^0, where $G^0 \subseteq GL(V)$, a classical linear group if G^0 is isomorphic to a classical group G under a *natural representation* of G.

1.2. Now in general (actually $m \geq 4$) $GL(V)$ contains many connected Lie subgroups G^0, other than just classical linear ones, which act irreducibly on V. In this paper we shall characterize among all such groups those which are classical linear. The result, although simply stated in terms of the group G^0 itself, is even simpler to state in terms of the Lie algebra $\mathfrak{g}^0$ of G. In this case the characterization involves only the existence of a single non-nilpotent element of $\mathfrak{g}^0$ of sufficiently low rank. Our main result (Theorem 2.5) yields the following corollary:

Corollary. *Let $\mathfrak{g}^0$ be a complex Lie algebra of linear transformations acting irreducibly on the m-dimensional complex vector space V.*

(1) *Then $\mathfrak{g}^0$ is the Lie algebra of all linear transformations on V if and only if it contains a non-nilpotent operator A of rank* 1.

(2) *If $\mathfrak{g}^0$ leaves invariant the non-zero bilinear form* $\mathbf{B}$ *on V, then $\mathfrak{g}^0$ is the Lie algebra of all linear transformations leaving* $\mathbf{B}$ *invariant if and only if $\mathfrak{g}^0$ contains a non-nilpotent operator of rank* 2.

Remark 1. As usual the invariance of a bilinear form $\mathbf{B}$ under the action of a Lie algebra $\mathfrak{g}^0$ is taken to mean $(Au, v) + (u, Av) = 0$ for all $u, v \in V$, $A \in \mathfrak{g}^0$.

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