SOME PROPERTIES OF COMPACTIFICATIONS

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Introduction. A *compactification* of a topological space X is a compact (Hausdorff) space containing a dense subspace homeomorphic with X. Since only completely regular spaces have compactifications, all spaces mentioned here will be completely regular unless the contrary is assumed explicitly. This paper is a study of properties of the sets of points which may be added to a space in compactifying it. We find several properties $\mathcal{P}$ such that for all spaces X

(*) *if the complement of* X *in* one *of its compactifications has property* $\mathcal{P}$, *then the complement of* X *in* any *of its compactifications has property* $\mathcal{P}$.

A list of such properties is provided in Theorem 2.2. It includes compactness, local compactness, σ-compactness, the Lindelöf property, and paracompactness.

Recall Čech's result [3] that any compactification AX of X is a continuous image of the Stone-Čech compactification βX of X under a mapping which takes $\beta X - X$ onto $AX - X$. The essence of the reason that (*) holds for the listed properties is that the restriction of this mapping to $\beta X - X$ preserves these properties in the strong sense that the domain has the property if and only if the range does. Indeed, this latter mapping is an example of what we call a *meshing map*; namely a mapping f of a space X onto a space Y which has an extension $\bar{f}$ over some compactification AX of X onto some compactification BY of Y, which maps $AX - X$ homeomorphically onto $BY - Y$. We call a property $\mathcal{P}$ a *meshing property* if whenever $f: X \rightarrow Y$ is a meshing map, then X has property $\mathcal{P}$ if and only if Y has property $\mathcal{P}$. Then a necessary and sufficient condition for (*) to hold for a property $\mathcal{P}$ is that $\mathcal{P}$ be a meshing property (Theorem 2.6).

All of the properties listed in our first paragraph are actually preserved by a wider class of mappings, namely those mappings $f: X \rightarrow Y$ such that f maps X continuously onto Y, f is closed, and for each $y \in Y$, the set $f^{-1}(y)$ is compact. We call these *fitting maps*, and the properties they preserve (in the strong sense given above), *fitting properties*. Every meshing map is fitting (and hence every fitting property is a meshing property); the converse is true for locally compact spaces, where these mappings coincide with those *proper* mappings in the sense of Leray [12] that are onto. Many fitting maps that are not meshing are provided by the fact that the projection map of the product of a non-locally compact space Y and a compact space containing at least two points onto Y fails to be meshing (Corollary 1.7). However, meshing properties that are not fitting are harder to find. We have identified one such (Example 2.1), but we have not found any among the more familiar topological properties.

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