# PROOF OF A CONJECTURE OF GROSSWALD

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1. **Introduction.** For positive integral $n$ let $a(n) = 2^{F(n)}$, where $F(n)$ is the total number of prime factors of $n$, repeated factors being counted in the appropriate multiplicity. Grosswald [1] and [2] proved that for positive real $x$ we have

$$\sum_{n \leq x}{}' a(n) = A_1 x \log x + B_1 x + O(x^c), \tag{1}$$

where $c$ is a constant between 5/6 and 1 and where $\sum'$ denotes summation over the odd positive integers $n$. (Throughout this paper large Latin letters, with and without subscripts, stand for certain absolute real constants.) He conjectured that actually

$$\sum_{n \leq x}{}' a(n) = A_1 x \log x + B_1 x + O(x^{\log 2/\log 3}), \tag{2}$$

which would be a best possible result—since the left-hand side has a discontinuity of saltus $x^{\log 2/\log 3}$ when $x$ is a positive integral power of 3.

The purpose of this note is to point out that (2) may actually be proved very simply and that similar results with still better error terms may be easily obtained for $\sum_{n \leq x}^{(k)} a(n)$, where $k \geq 2$ and $\sum^{(k)}$ denotes summation over the positive integers $n$ not divisible by any of the first $k$ primes. These results are given in formulas (13) and (18) below. In deriving such formulas it is not necessary to calculate the coefficients in the main terms explicitly in the course of the proof (although that can certainly be done), since they can easily be determined subsequently by trivial function-theoretic considerations, as we shall explain in detail below. (See formulas (19)–(21) for the values of these coefficients in the present situation.)

Using the obvious formula

$$\sum_{n \leq x} a(n) = \sum_{\mu=0}^{[\log x/\log 2]} 2^{\mu} \sum_{n \leq 2^{-\mu} x}{}' a(n),$$

Grosswald deduced in a few lines from (1) that

$$\sum_{n \leq x} a(n) = Ax \log^2 x + Bx \log x + O(x), \tag{3}$$

where the summation is now over all positive integers not exceeding $x$. This is, of course, a best possible result too, since the left-hand side has a discontinuity of saltus $x$ when $x$ is a positive integral power of 2. (Grosswald actually claimed to have proved a sharpened version of (3) with an error term $O(x^c)$, but he inadvertently omitted a factor in his computation of the error term and, in fact, his argument proves only (3).)

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