HÖLDER CONTINUITY AND NON-LINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Part I of this paper deals with estimates for the degree of continuity of mappings $(x, y) \to (p, q)$ subject to inequalities of the form $p_x^2 + p_y^2 + q_x^2 + q_y^2 \leq$ constant $\partial(p, q)/\partial(x, y)$ + Constant. The results obtained are applied in Part II to the question of the Hölder continuity of second order derivatives of a solution of a second order, elliptic partial differential equation in two independent variables. In particular, a crude (but useful) form of E. Hopf's [3] interior estimates (for the case of two independent variables) results very simply, without the use of potential theory.

PART I

1. Let $p(x, y)$, $q(x, y)$ be functions, say, of class C^1 on a domain D with partial derivatives satisfying the inequality

$$p_x^2 + p_y^2 + q_x^2 + q_y^2 \leq \sigma\partial(q, p)/\partial(x, y) + \tau, \tag{1}$$

where σ, τ are positive constants. This part of the paper is concerned with *a priori* estimates for the degree of continuity of p, q which depend only on σ, τ (and D).

If $\sigma < 2$, then p, q satisfy uniform Lipschitz conditions (with a Lipschitz constant depending only on σ and τ). This can be deduced from the fact that (1) can be written, for example, as

$$(p_x + \tfrac{1}{2}\sigma q_y)^2 + (p_y - \tfrac{1}{2}\sigma q_x)^2 + (1 - \sigma^2/4)(q_x^2 + q_y^2) \leq \tau.$$

It will be assumed below that $\sigma > 2$.

If $\tau = 0$ in (1), then (1) implies that the mapping $(x, y) \to (p, q)$ is "quasi-conformal". In this case, it is known (cf. Morrey [4]) that, on every compact subset D' of D, p, q are Hölder continuous of order λ, where

$$0 < \lambda < 1, \lambda + \lambda^{-1} = \sigma \qquad (\sigma > 2), \tag{2}$$

with a Hölder constant depending only on σ and D'. A similar result (Nirenberg [5]) is known to hold when $\tau > 0$, but the Hölder exponent is given only as $1/\sigma$ in [5]. The disadvantage of this last estimate is that the Hölder exponent $1/\sigma$ satisfies $1/\sigma < \frac{1}{2}$ (in contrast to the fact that λ in (2) satisfies $\lambda \to 1$ as $\sigma \to 2$).

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