STIRLING NUMBERS OF THE SECOND KIND

By L. Moser and M. Wyman

1. Introduction and summary. In the notation of finite differences the Stirling numbers of the second kind are given by

$$\sigma_n^m = \Delta^m 0^n / m!. \tag{1.1}$$

This is equivalent to

$$\sigma_n^m = \frac{(-1)^m}{m!} \sum_{k=1}^{m} (-1)^k \binom{m}{k} k^n. \tag{1.2}$$

These numbers play an important role in the calculus of finite differences, in combinatorial problems of statistics (see for example W. L. Stevens 1937), and in other branches of mathematics. An outline of the elementary properties of the Stirling numbers of the second kind is given by C. Jordan (1947). He expresses the opinion that these numbers are as important as the Bernoulli numbers and makes them occupy a central position in his development of the theory of finite differences. The object of the present paper is to develop asymptotic expansions for σ_n^m which are valid for a much larger range of values than have hitherto been covered.

The attempts to obtain manageable formulae for σ_n^m with m and n large go back to Laplace (1814) who derived an approximate formula for the related numbers $\Delta^n 0^i / n^i$. His formula was criticized by Tait and by Cayley (1887) who developed another approximation. In recent times Jordan (1947) has given the formulae

$$\sigma_n^m \sim m^n / m!, \tag{1.3}$$

and

$$\sigma_n^{n-m} \sim n^{2m} / m! 2^m. \tag{1.4}$$

He points out, however, that these formulae are acceptable only for m quite small in comparison with n and that the error increases rapidly as m increases. For the case $n = 50$ the following short table illustrates this fact.

m	$m^{50}/m!$	σ_{50}^m	$50^{2m}/(m!)2^m$	σ_{50}^{50-m}
5	7.4015×10^{32}	7.4010×10^{32}	2.5431×10^{13}	1.3133×10^{13}
10	2.7557×10^{43}	2.6155×10^{43}	2.5665×10^{24}	1.4938×10^{23}
15	4.8760×10^{46}	2.9226×10^{46}	2.1685×10^{34}	2.4236×10^{31}

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