THE TRANSFORMATION FORMULA FOR THE DEDEKIND MODULAR FUNCTION AND RELATED FUNCTIONAL EQUATIONS

By Shô Iseki

Introduction. The Dedekind modular function $\eta(\tau)$, defined by

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi in\tau}) \qquad (\Im(\tau) > 0),$$

plays an important role in the theory of elliptic modular functions, and is also closely connected with the generating function for $p(n)$, the number of unrestricted partitions of a positive integer n (see Rademacher [5]).

The celebrated transformation formula for $\eta(\tau)$, which is concerned with a modular substitution of τ, reads

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon\left(\frac{c\tau + d}{i}\right)^{\frac{1}{2}} \eta(\tau), \tag{1}$$

where a, b, c and d are rational integers satisfying $ad - bc = 1$, $c \geq 0$ (the case $c = 0$ is rather trivial, and we shall assume $c > 0$ in the following); the square root means the principal branch, and ϵ is a certain 24th root of unity depending on a, b, c and d.

Actually, there exists a precise transformation formula for $\log \eta(\tau)$ which may be used to determine the exact value of ϵ in (1). (For a direct proof of (1) with an explicit form of ϵ, not depending on logarithms, see W. Fischer [2].)

We can write, since $ad - bc = 1$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} H & -\dfrac{hH + 1}{k} \\ k & -h \end{bmatrix}$$

where h and k (> 0) are relatively prime integers and H is an integer such that $hH \equiv -1 \pmod{k}$. Putting $(c\tau + d)/i = z$ $(\Re(z) > 0)$, we find that

$$\tau = \frac{iz + h}{k}, \qquad \frac{a\tau + b}{c\tau + d} = \frac{i/z + H}{k}.$$

With these new characters the transformation formula for $\log \eta(\tau)$ is written in the form

$$\sum_{n=1}^{\infty} \lambda\left(\frac{n}{k}(z - ih)\right) + \frac{\pi}{12k}\left(z - \frac{1}{z}\right) = \sum_{n=1}^{\infty} \lambda\left(\frac{n}{k}\left(\frac{1}{z} - iH\right)\right) + \frac{1}{2}\log z + \pi is(h, k), \tag{2}$$

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