ASYMPTOTIC SOLUTIONS OF OSCILLATORY INITIAL VALUE PROBLEMS

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Introduction. In this paper we investigate the asymptotic behavior of solutions of linear partial differential equations with highly oscillatory initial values. We consider an operator M (a first order matrix operator throughout this paper), and a solution $u(x^1, \cdots, x^m, t) = u(x, t)$ of the equation

$$Mu = 0$$

with initial values

$$u(x, 0) = \phi(x)$$

where $\phi(x)$ depends on a parameter ξ in an oscillatory manner

$$\phi(x) = e^{i\xi l(x)}\psi(x),$$

with $l(x)$ some given real-valued function.

In §1, we shall construct a formal asymptotic series for u:

$$u \sim e^{i\xi l(x,t)}\left\{v_0 + \frac{1}{\xi}v_1 + \cdots\right\}$$

and show that, if the hyperplane $t = 0$ is space-like for the operator M, our expansion has an asymptotic validity. This is potentially useful in calculating asymptotic diffraction patterns (see [10], [11], and [14] for a detailed discussion); namely, the problem of diffraction can be formulated as finding the steady state of a solution with oscillatory initial values.

In §2 we use the formal asymptotic series to show that, if the hyperplane $t = 0$ is *not* space-like for an operator M with analytic coefficients, then the initial value problem is incorrectly posed. The method used in this paper is a generalization of the method which Hadamard used to obtain this result for operators with constant coefficients.

In §3, we study the manner in which solutions of hyperbolic equations depend on their initial values. This is done by using the asymptotic expansion to construct, in §4, by Fourier synthesis, an approximate influence function; i.e. we find—modulo a smooth function—the kernel G of the integral operator expressing the value of a solution of a hyperbolic equation at any point P in terms of its values at points Q of the initial manifold. A different construction of the influence function, based on an expression of the δ-function in terms of

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