A GENERALIZED SCHWARZIAN DERIVATIVE AND CONVEX FUNCTIONS

By R. F. Gabriel

1. **Introduction.** In an earlier paper [1] the author showed that a bound on the Schwarzian derivative of $f(z) = z^{-1} + \cdots$ was sufficient for the convexity of the function. This paper will introduce a generalization of the Schwarzian derivative and present an application of it to multivalent functions of the form

$$f(z) = z^{-n} + \cdots \tag{1.1}$$

which are analytic and single-valued in $0 < | z | < 1$.

The expression

$$\{ f(z), z\}_n \equiv \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{n+1}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{1.2}$$

will be called a generalized Schwarzian derivative.

The principal result is the following:

Theorem I. *Let $f(z) = z^{-n} + \cdots$ be analytic and single-valued in $0 < | z | < 1$ with $f'(z) \neq 0$ in $0 < | z | < 1$, and let*

$$| \{ f(z), z\}_n | \leq \frac{(n+1)c_n}{| z |} \quad for \quad 0 < | z | < 1 \tag{1.3}$$

where c_n is defined in (5.11). *Then $f(z)$ maps $| z | = r < 1$ onto a curve which is convex of order $-n$. If, further, there is a value w_0 for which $f(z) \neq w_0$ in $0 < | z | < 1$, then $f(z)$ is n-valent and starlike with respect to w_0 in $0 < | z | < 1$. For each n the constant c_n is the best possible one.*

2. **Preliminaries.**

Lemma 2.1. *Let $f(z) = h(z)/z^n$ where $h(z)$ is analytic and single-valued in $| z | < 1$, $h(0) = 1$, $h'(0) \neq 0$, and n is a positive integer greater than* 1. *Then $\{f(z), z\}_n$ has a simple pole at the origin.*

For $f(z)$ defined above we have

$$f'(z) = \frac{zh'(z) - nh(z)}{z^{n+1}}. \tag{2.1}$$

Taking the logarithm of (2.1) and differentiating again we obtain

$$\frac{f''(z)}{f'(z)} = \frac{zh''(z) - (n-1)h'(z)}{zh'(z) - nh(z)} - \frac{n+1}{z}. \tag{2.2}$$

Received January 4, 1957; in revised form, June 26, 1957.