HÖLDER GROUPS

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According to a long established theorem of Hölder, any Archimedean linearly ordered group is isomorphic to a subgroup of the additive group of real numbers. In particular, such a group is an Abelian group.

There are several ways of generalizing this theorem. The usual procedure is to assume a partial order rather than a linear order, and to require of the group a certain behavior relative to the partial order. For example, the existence of order units is frequently assumed. The additive group of real numbers is replaced by a suitable generalization, say by a function space or by a partially ordered vector space of a more general nature.

The purpose of this note is to offer another sort of generalization. In this approach, all *a priori* reference to ordering is suppressed. This casts the entire burden of the argument on the group structure. The assumption of an ordering in the group is replaced by a maximality condition, and the more restrictive hypothesis of Archimedean type is replaced by a correspondingly stronger maximality condition. The ordering also disappears from the groups used in the representation theory. In their place appear locally convex linear spaces.

The development in this note is based on an earlier paper by the author [3]. The present paper begins by extending some of the concepts of this previous work to non-Abelian groups. The exposition of this extension is somewhat condensed. Since the lack of commutativity offers no essential difficulty, no attempt is made to repeat arguments which need only slight alterations to make them applicable in the non-Abelian case.

1. **The radical of a non-Abelian group.** Let G be a topological group, in which the group operation is written additively. As usual, the Hausdorff separation axiom is assumed, and the continuity of inversion as well as the joint continuity of group addition is assumed. The group is not in general Abelian, despite the use of the additive notation.

As in [3], we define, for any non-empty subset A of G, the set $s(A) = \{x \in G: (x + A) \cup (A + x) \subset A\}$. A subset A of G is therefore a semigroup in G if and only if $A \subset s(A)$. For any subset A, the set $s(A)$ is a semigroup in G.

Definition 1. A subset A of a group G is called a normal set if A is invariant under all the inner automorphisms of G.

Thus A is normal if and only if $A + x = x + A$ for each $x \in G$. It follows therefore that if A is a normal set, then $x \in s(A)$ if and only if $x + A \subset A$. For a normal set A, the set $s(A)$ is a normal semigroup, and therefore the set $b(A) = s(A) \cap s(-A)$ is a normal subgroup of G.

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