THE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS. III

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1. **Introduction.** In this paper we consider the nonlinear differential equation

$$x'' + p(t)x + 2q(t)x^3 = F(t) \tag{1}$$

in which the coefficients of the restoring force as well as the external force depend on the time t.

In the sequel, $p(t)$, $q(t)$ and $F(t)$ are assumed to be real-valued, bounded, Lebesgue-measurable functions of the real variable t and periodic with a common period L. It is further assumed that $p(t)$ and $q(t)$ are even functions possessing positive lower bounds and $F(t)$ is odd, positive and possessing a non-negative lower bound on the open interval $(0, L/2)$. We shall use p_1, q_1 and F_1 to denote the greatest lower bounds of p, q, and F respectively on $(0, L/2)$ and p_2, q_2 and F_2 their least upper bounds on $(0, L/2)$. (It is sufficient to require these bounds to hold almost everywhere on $(0, L/2)$.) We note that all these bounds are positive, by assumption, save possibly F_1 which is non-negative.

Several criteria for the existence of periodic solutions, harmonic as well as subharmonic of order n, are obtained and stated in Theorems 1–4. These criteria are expressed in terms of the period L and of the bounds of p, q and F and become very simple when q or F are small perturbations. The oscillatory behavior of the periodic solutions are also described in terms of the number of zeros of the solutions and of their derivatives over a half period. The general oscillatory properties of the solutions are first investigated; then they are used to obtain the criteria for the existence of periodic solutions. This approach was suggested by the Classical Sturmian method (Chap. 8 in [2], [1]). In two previous papers [5], [3], we used a similar approach to study two special cases of the equation (1), namely $F = 0$, $p > 0$ and $q > 0$ in [5] and $F = 0$, $p > 0$ and $q < 0$ in [3], which can be considered as the differential equations of hard spring and soft spring respectively. This paper does not use the results in [5] and [3].

Equation (1) is a Duffing's equation with periodic coefficients. The constant coefficient case is discussed at some length in [4].

By a solution of (1) in an interval I we mean a function $x(t)$ which is absolutely continuous together with x' on I and satisfies (1) almost everywhere on I (i.e. in the sense of Carathéodory). For the uniqueness and existence of solutions and their general properties, see Chap. I in [2], especially Theorems 3.1 and 7.1 which hold for equation (1) even though p, q and F are only bounded measurable functions.

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