# A PROPERTY OF INTEGRAL MEANS

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The theory of ordinary convergence of singular integrals is fully developed. However, whether or not the convergence holds with respect to a given norm has only been established in special cases.

An example is the space $L_p(0, 1)$, $p \geq 1$. If $\{k_n(t)\}$ is a sequence of functions which satisfies conditions such as

a) $\lim_{n\to\infty} \int_{-\infty}^{\infty} k_n(t)\, dt = 1$, b) $\int_{-\infty}^{\infty} | k_n(t) | \, dt < M$ for fixed $M$ and all $n$, and c) $\lim_{n\to\infty} \int_{|t|<\delta} | k_n(t) | \, dt = 0$, for every $\delta > 0$, then the sequence $x_n(t) = \int_0^1 k_n(t - u)x(u)\, du$, $n = 1, 2, \cdots$, converges in $L_p$ to $x(t)$.

The proof of this fact, [4], leans heavily on the Hölder inequality. It may accordingly be extended to other spaces such as those of Orlicz, [6], and Lorentz, [5], for which there are analogous inequalities.

S. Bochner, [2], has stated as a heuristic principle that a Banach space $X$ of summable functions has the property stated above for $L_p$ if $x(t) \in X$ implies $x(t + u) \in X$, for every $u$, $|| x(t + u) || = || x(t) ||$, and $x(t + u)$ is a continuous function of $u$ in $X$.

This statement is not true in full generality, as the following example shows.

Let $x(t)$ be defined and summable on the set $[0, 1)$ of real numbers modulo 1, let $\varphi(t)$ be an indefinite integral of $x(t)$, and suppose that the translations $\varphi(t + u)$, $0 \leq u < 1$, of $\varphi(t)$ form a linearly independent set of functions. For example

$$\begin{aligned} x(t) &= 1, \qquad \tfrac{1}{4} < t < \tfrac{3}{4}, \\ &= 0, \qquad 0 \leq t < \tfrac{1}{4} \text{ and } \tfrac{3}{4} \leq t < 1, \end{aligned}$$

has this property.

Let $\{h_n\}$ be a sequence of positive numbers, converging to 0, which are linearly independent over the rationals.

Now, for every $h > 0$, let

$$x^h(t) = \frac{1}{h} \int_0^h x(t + u)\, du = \frac{1}{h} [\varphi(t + h) - \varphi(t)].$$

We show that the set

$$A = [x^{h_n}(t + u)], \qquad 0 \leq u < 1, \qquad n = 1, 2, \cdots,$$

is composed of linearly independent functions. Suppose, on the contrary, that

$$\sum_{i=1}^{n} \frac{c_i}{h_i} [\varphi(t + h_i + k_i) - \varphi(t + k_i)] = 0,$$

Received February 4, 1957. This note is related to work sponsored by National Science Foundation grant NSF G 2267 on ordered systems.