TWO PARAMETER MOMENT PROBLEMS

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1.1. In a celebrated memoir [11], published in 1894–95, T. J. Stieltjes wrote down a necessary and sufficient condition that a sequence of constants $\{\mu(n); n = 0, 1, \cdots\}$ may be written as the integral

$$\mu(n) = \int_0^\infty x^n \, d\alpha(x), \tag{1}$$

where the measure $d\alpha(x)$ is non-negative and bounded. This condition is that for any finite set $\{\zeta_i\}$ of real numbers, we have

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \zeta_i \zeta_j \mu(i + j) \geq 0 \quad \text{and} \quad \sum_{i=0}^{n} \sum_{j=0}^{n} \zeta_i \zeta_j \mu(i + j + 1) \geq 0. \tag{2}$$

The next important advance in the representation problem was due to H. L. Hamburger [5] who, in 1920–21, extended the domain of integration in (1) to the whole real axis. His result was as follows: A necessary and sufficient condition that there exists a non-negative bounded measure $d\alpha(x)$ such that

$$\mu(n) = \int_{-\infty}^\infty x^n \, d\alpha(x) \tag{3}$$

is that for any finite set $\{\zeta_i\}$ of real numbers,

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \zeta_i \zeta_j \mu(i + j) \geq 0. \tag{4}$$

In 1921, F. Hausdorff [7] gave conditions that a sequence $\{\mu(n)\}$ may be represented by a moment integral, where the domain of integration is a finite interval. His conditions may be translated to read as follows: A necessary and sufficient condition that there exists a bounded non-negative measure $d\alpha(x)$ such that

$$\mu(n) = \int_0^1 x^n \, d\alpha(x) \tag{5}$$

is that for any finite set $\{\zeta_i\}$ of real numbers,

$$0 \leq \sum_{i=0}^{n} \sum_{j=0}^{n} \zeta_i \zeta_j \mu(i + j + 1) \leq \sum_{i=0}^{n} \sum_{j=0}^{n} \zeta_i \zeta_j \mu(i + j). \tag{6}$$

This result was extended to higher dimensions by Hildebrandt and Schoenberg [10] in 1933.

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