THE TEMPORAL BEHAVIOUR OF A WAVE PACKET

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In this note we consider trigonometric integrals of the form

$$F(x, t) = \int f(\alpha) \exp [i(\alpha x + g(\alpha)t)] \, d\alpha \tag{1}$$

where $x = (x_1, \cdots, x_n)$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, with x and α, as well as t, real, $\alpha x = \sum \alpha_\nu x_\nu$, $d\alpha = d\alpha_1 \cdots d\alpha_n$, and where the integration extends through the entire α-space. The function $g(\alpha)$ is supposed to be real-valued. We prove the following

(1) Theorem. *Let f be Lebesgue-integrable in α-space. Suppose that the α-space is the union of three sets with the following properties, respectively*: a) $f = 0$, b) *has measure zero,* c) *open set with $g(\alpha)$ of class C'' therein, and with the second derivatives $g_{,\nu\mu}$ never zero simultaneously. Then*

$$\sup_x | F(x, t) | \to 0 \quad \text{as} \quad t \to \infty . \tag{2}$$

This theorem has the following immediate consequence:

(2) Theorem. *Let f be Lebesgue-integrable and let $g(\alpha)$ be analytic throughout α-space. Then either* (2) *holds or $F = F(x - ct) \exp (iat)$, where a and c are real constants, c being a vector.*

In fact, an analytic function in α-space vanishes either everywhere or only in a subset of measure zero. This implies that either g is linear or that the points α where all the second derivatives vanish form a set of measure zero. In the first case we find that $g = -\sum \alpha_\nu c_\nu + a$, $F = F(x - ct)$ times $\exp (iat)$, whereas in the second case Theorem 1 applies.

In physical terms, (1) represents a wave packet with grad g as the vector of the group velocity. Under the condition of analyticity, as expressed in Theorem 2, the group velocity is either constant or not. In the first case, the wave packet travels essentially as a rigid whole, whereas, in the second case, the waves are dispersed and the resultant (1) dies down uniformly in x-space.

The actual core of Theorem 1 is its one-dimensional case. The author proved the theorem when he found that he needed it for the discussion of the stability properties of the solutions of a certain space-time integro-differential system (see [1], the fourth lecture). We start by proving the one-dimensional case of the theorem.

(3) Theorem. $n = 1$. *The hypotheses are the same as in the first theorem, with the difference that the set* c) *is the union of a sequence of α-intervals in each*

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