INTERPOLATION IN POLYNOMIAL CLASSES AND MARKOFF'S INEQUALITY

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1. **Introduction.** The interpolation theorem of M. Riesz [3] is widely used in determining the bounds of operators defined on L_p spaces. For its application, however, the interpolation theorem requires that the operator in question have as domain of definition a subset dense in all L_p spaces (e.g. the "simple functions").

For many reasons, however, it would be desirable to reach substantially the conclusions of the Riesz theorem with operators defined on much more restricted linear subspaces. As an example of this situation we cite the operation of differentiation defined on trigonometric or power polynomials of degree n, for some fixed n.

The aim of this paper is to throw some light on the problem. We shall interpolate operations defined on the "polynomials of order n" of some rather general orthogonal developments, (those which we call "regular").

The method of proof of the interpolation theorem depends heavily on the notion of "generalized delayed means." For the Fourier expansion of a function f, the "delayed means" were introduced a long time ago.

We shall need an appropriate generalization of this notion, which we present in §2. The notion of delayed means has previously been generalized in a particular case by Zygmund [5].

In §§3 and 4 we shall apply the interpolation theorem in proving inequalities related to and extending Markoff's inequality for power polynomials.

2. **General interpolation theorem.** Let us be given an orthonormal system $\{\phi_k(x)\}$, $k = 0, 1, 2, \cdots$ over a measure space E with measure $d\nu$. For any $f \varepsilon L_2\ (E, d\nu)$ we form $a_k = \int_E f\bar{\phi}_k d\nu$, and define $s_n(f) = \sum_0^n a_k\phi_k(x)$, the n-th sum of the expansion of f. We may also define, in the usual manner, $\sigma_n^{(r)}(f)$ [or simply $\sigma_n^{(r)}$], the Cesàro means of order r of $s_n(f)$. Let us also call any sum of the form $\sum_0^n a_k\phi_k(x)$ a *polynomial of degree n.*

The following theorem characterizes the notion of "generalized delayed means".

THEOREM 1. *Let r be non-negative and integral. There exist $r + 1$ parameters $\alpha_1^r, \alpha_2^r, \cdots \alpha_{r+1}^r$ (depending on n) which are uniformly bounded: $|\ \alpha_i^r\ | \leq A$, A independent of n. We can also find a fixed integer N, so that the following holds:*

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