ONE-PARAMETER SEMI-GROUPS OF MAPS

By George Hufford

A one-parameter semi-group of maps on a topological space E is a set $\{u_t\}$ of maps from E into itself which is parametrised by the non-negative real numbers, and which satisfies the property that for all $s, t \geq 0$

$$u_s \circ u_t = u_{s+t} , \tag{0.1}$$

where by the circle we mean composition; i.e., for each $x \in E$, $(f \circ g)(x) = f(g(x))$. It is convenient to require that u_0 be the identity map, and ordinarily we shall want u_t continuous and $u_t(x)$ continuous as a map from $E \times [0, \infty)$ to E in some sense or other.

Our efforts here are toward describing what might be called an infinitesimal generator of such a semi-group. To be precise, consider the Banach algebra $\mathfrak{F}$ of bounded, real-valued functions defined on E, the norm being given for any $f \in \mathfrak{F}$ by $\| f \| = \sup | f(x)|$. Now for each non-negative t and $f \in \mathfrak{F}$ put

$$T_t f = f \circ u_t . \tag{0.2}$$

Then the set $\{T_t\}$ is a one-parameter semi-group of linear transformations on $\mathfrak{F}$ with the further properties that $T_0 = I$ and $\| T_t \| = 1$. If we can prove the strong continuity of $\{T_t\}$ or of $\{T_t\}$ restricted to some invariant closed subspace of $\mathfrak{F}$, then we are in a position to use the Hille-Yosida theorem [3; 238; 4]. In particular, we can speak of the infinitesimal generator of $\{T_t\}$, and, as we shall show, this infinitesimal generator defines not only $\{T_t\}$ but also the original semi-group $\{u_t\}$.

A semi-group of linear transformations on a function space which can be expressed in the form (0.2) we shall call a *translation semi-group,* and we shall say that it is *associated* with the semi-group of maps $\{u_t\}$.

If the range of the parameter t consists of the entire real line, then we have a one-parameter group of maps. Most of the statements we shall make about semi-groups apply equally well to groups with slight modifications that will be left to the reader. The basic theorem here [see 3; 322], corresponding to the Hille-Yosida theorem for semi-groups, states that the resolvent J_λ exists also for negative real λ and $\| J_\lambda \| \leq 1/| \lambda |$.

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1. **Compact spaces.** First we shall consider the case where E is compact (i.e., bi-compact and Hausdorff). Here we can make use of a theorem of Gelfand

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