STRUCTS ON THE 1-SPHERE

By I. S. Krule

The purpose of this paper is to establish some properties of binary relations on Hausdorff spaces. Studies of similar nature have been made by a number of other authors, in particular Nachbin [4], Strother [5], Wallace [6; 7], and Ward [9; 10; 11], and this paper may be regarded as a continuation of their work.

If "$<$" is any transitive binary relation on a set X, then the elements x of X may be separated into equivalence classes $[x]$ by letting $[x]$ be x together with the set of all $y \varepsilon X$ for which $x < y$ and $y < x$. If X is a topological space, then the set Y of equivalence classes may be topologized by calling open those subsets of Y whose pre-images are open under the canonical function $x \rightarrow [x]$. Under certain conditions the sets $[x]$ form an upper semi-continuous decomposition of X, and the decomposition space Y will reflect properties of X with respect to both the relation "$<$" and the topology on X.

We are concerned here with the topological nature of Y when "$<$" is a *struct on* X (definition below), and our main result, Theorem 5, is a characterization of the class of boundary curves which are cyclic chains as the class of decomposition spaces of the simple closed curve ($=S^1$) over the elements of the class of upper semi-continuous decompositions of S^1 resulting, as indicated above, from certain structs on S^1. It is known (see Whyburn [12]) that every image of S^1 under a non-alternating map (= continuous function) is a boundary curve, though not necessarily a cyclic chain, and that each boundary curve is the image of S^1 under a light non-alternating map. We show (Theorem 3) that if F is any map of S^1 with the property that the set of images of points under F^{-1} is the upper semi-continuous decomposition of S^1 determined by a certain type of struct on S^1, then F is non-alternating and $F(S^1)$ is a cyclic chain as well as a boundary curve; further, we show (Theorem 4) that if Y is any boundary curve which is a cyclic chain, then there is a light non-alternating map F of S^1 onto Y enjoying the above property.

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Throughout this paper X is a Hausdorff space, and "continuum" *means* "compact connected Hausdorff space."

We recall the following definitions from [7]. A *struct on* X is a non-empty closed transitive subset of $X \times X$. If L is a struct on X, if $a \varepsilon X$, and if $A \subset X$, then

(1) $L(a) = \{x \varepsilon X \mid (x, a) \varepsilon L\}$;
(2) $L(A) = U\,\{L(x) \mid x \varepsilon A\}$;

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