MINIMIZING TRANSFORMATIONS OF HERMITIAN FUNCTIONALS, AND PRODUCT INTEGRATION

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1. **Introduction.** Let A and B be inner products (i.e., positive definite Hermitian bilinear functionals) for two real or complex Hilbert spaces with the same underlying linear space H. Among all linear transformations T of H onto itself taking A into B (in the sense that $A(Tu, Tv) = B(u, v)$ for all u, v in H), certain transformations are distinguished by the fact that they are as close to the identity transformation I as possible, i.e. that the magnitude (in some suitable sense) of $T - I$ is as small as possible. It was shown by Loewner [2] (see Theorem 1 below) that there is a unique such minimizing transformation if closeness of transformations is measured by means of the *norm* based on A or on B, if A and B are not too far apart. A corresponding theorem (without uniqueness, however) is presented here (Theorem 2) in which the *bound* based on A or on B is used for measuring closeness. A more general problem is that of finding a minimizing transformation of A into B via a path in the space of complete inner products on H. This leads to a kind of product integration, which is carried out in Theorems 3 through 5. These theorems are special cases of lemmas on a type of product integration which is slightly more general than that of Volterra [9], but less general than that of Stewart [7].

2. **Notation.** Throughout, H will be a fixed real or complex linear space, and E will be a fixed *complete* inner product for H. The E-length $| u |_E$ of a vector u in H is defined as $(E(u, u))^{\frac{1}{2}}$. The unit E-sphere, consisting of all vectors of E-length 1, is denoted by SE. If T is any linear transformation (of all of H into itself—this always understood), then the E-bound of T, written as $| T |_E$, is defined as $\sup_{u \varepsilon SE} | Tu |_E$, and is also equal to $\sup_{u \varepsilon SE,\, v \varepsilon SE} | E(u, Tv)|$. The E-norm $| T |_E'$ of T is defined as $(\sum_{\phi \varepsilon \Phi} | T\phi |_E^2)^{\frac{1}{2}}$, or as $(\sum_{\phi \varepsilon \Phi, \psi \varepsilon \Phi} | E(\phi, T\psi)|^2)^{\frac{1}{2}}$, where Φ is any complete E-orthonormal set for H, [5; 66]. If E_1 is any bilinear functional with finite E-bound

$$| E_1 |_E = \sup_{u \varepsilon SE, v \varepsilon SE} | E_1(u, v)|,$$

then $E^{-1}E_1$ will denote the uniquely determined linear transformation such that $E(u, (E^{-1}E_1)v) = E_1(u, v)$ for all u, v in H. The convention, in the case

Received November 16, 1956. Presented to the American Mathematical Society, November 25, 1955, under the title "Transformations of Hermitian functionals characterized by an extremum property".

The author wishes to thank Professor Charles Loewner for his kind and stimulating guidance. Some of the material in this paper is taken from the author's Ph. D. dissertation, Syracuse University, 1953. Portions of the research for this paper had ONR support at Stanford University.