A NEW PROOF OF THEOREMS OF PERRON AND FROBENIUS ON NON-NEGATIVE MATRICES: I. POSITIVE MATRICES

By Alfred Brauer

A well known theorem of O. Perron [13] and G. Frobenius [7] states that a positive matrix A has a positive characteristic root ω which is simple and greater than the moduli of all the other characteristic roots of A. The coordinates of a characteristic vector belonging to ω can be chosen as positive numbers.

Perron's proof uses limits; Frobenius' proof of this theorem and its generalization for non-negative matrices [8] is purely algebraic. Using the Brouwer fixed point theorem P. Alexandroff and H. Hopf [1; 480–481] gave a topological proof that a non-negative matrix has at least one non-negative root. This method was extended by G. Debreu and I. N. Herstein [5] to prove the other parts of the theorems of Frobenius. Recently, two other interesting, but not purely algebraic proofs of the theorems of Frobenius, were given by H. Wielandt [15] and A. S. Householder [9].

In this paper a new proof for the main results of Frobenius will be given. This proof has not the advantage of being purely algebraic since the continuity of the roots of algebraic equations is used. But its principle is very simple.

Moreover, not only the existence of a greatest non-negative characteristic root ω of a non-negative matrix A is proved, but the proof gives also a simple method to compute ω and the coordinates of a characteristic vector belonging to ω as exactly as needed without determining the characteristic equation. By multiplying the rows of A successively by certain constants and dividing the corresponding columns by the same constants, a sequence of matrices similar to A is obtained. If $R^{(\nu)}$ denotes the maximum of the sums of the elements of each row of the ν-th of these matrices and $r^{(\nu)}$ its minimum, then the intervals $\{r^{(\nu)} \cdots R^{(\nu)}\}$ form a nested set which converges to ω. At each step the length of the interval gives an upper bound for the error.

A non-negative matrix is called stochastic if the sum of the elements in each row equals 1. In my paper [3], I called a matrix a generalized stochastic matrix if each row-sum equals s. R. v. Mises [10, 536] pointed out that the theorems of Frobenius can be used for stochastic matrices since they are special cases of non-negative matrices, and V. Romanovsky [14] formulated these results for stochastic matrices.

Conversely, it will be shown in this paper that the properties of the characteristic roots of arbitrary positive or non-negative matrices follow from those of generalized stochastic matrices. In particular, the theorems of Frobenius will

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