THE WIENER-HOPF EQUATION WHOSE KERNEL IS A PROBABILITY DENSITY

By Frank Spitzer

1. The equation in its simplest form is written

$$F(x) = \int_0^\infty k(x - y)F(y)\,dy, \qquad x > 0, \tag{1.1}$$

where $k(x)$ is a known function. The present study is motivated by results concerning a certain probability model (the maximum of successive partial sums of identically distributed independent random variables), which can be found in [11]. Therefore $k(x)$ is taken to be a probability density, while the solution of (1.1) which is of interest in this context must be a distribution function, and we shall so restrict what we call a solution. To be precise we shall say that $F(x)$ is a P-solution (P for probability) or a P^*-solution if it satisfies respectively conditions

(P) $F(x)$ is non-decreasing and continuous on the right, $F(x) = 0$ for $x < 0$ and $\lim_{x\to\infty} F(x) = 1$, or

(P^*) $F(x)$ is non-decreasing and continuous on the right, $F(x) = 0$ for $x < 0$ and $F(x)$ does not vanish everywhere.

As an example of well known results concerning P-solutions we mention

Theorem 1. *Let $k(x)$ be a probability density with finite first moment, i.e.*

$$k(x) \geq 0, \qquad \int_{-\infty}^{\infty} k(x)\,dx = 1, \qquad \int_{-\infty}^{\infty} |\, x \,|\, k(x)\,dx < \infty.$$

Then equation (1.1) *has either a unique P-solution or no P-solution at all, according as $\int_{-\infty}^{\infty} xk(x)\,dx < 0$ or ≥ 0.*

This result was obtained by D. V. Lindley [7], as an application of the strong law of large numbers. It implies a theorem in the theory of the one server queue which states that such a queue is ergodic if the expected interarrival time exceeds the expected service time. The solution $F(x)$ is then the limiting distribution of the waiting time of the n-th customer. A generalization of Theorem 1 to the case of the n-server queue was obtained by Kiefer and Wolfowitz [6], which involves more complicated equations than (1.1).

Our aim, in §2, is to find a condition on $k(x)$ which is both necessary and sufficient for a unique P-solution to exist, without assuming that $k(x)$ has a finite first moment. (A queue may be ergodic even if all moments are infinite.) This condition is given in Theorem 2, for a somewhat more general equation than (1.1). The proof, and the theory in later sections, makes essential use of

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