COLLECTIONS WHOSE SUMS ARE TWO-MANIFOLDS

By John Marshall Slye

The purpose of this paper is to establish conditions that are necessary and sufficient in order that an upper semicontinuous collection of arcs (or simple closed curves) filling up a locally connected metric space whose decomposition space is an arc fill a 2-cell (annulus, Möbius strip, or Klein bottle). It is sufficient for such a collection to be continuous and equicontinuous [1]. However, these conditions are not necessary. Also these are conditions restricting the collection filling the space. Here two conditions are used that restrict the space being filled rather than the collection filling the space. The first of these conditions is that no point (nor pair of points) separates the space. The second condition is that the space be unifoliate.

Definition 1. A space is *unifoliate* at a point P if and only if there exists an open set D containing P such that no closed set B of D irreducible with respect to separating two points of D contains two points P' and P'' such that $B - (P' + P'')$ is the sum of three mutually separated sets each of which has $P' + P''$ in its closure. If S is unifoliate at each of its points, then S is unifoliate.

Every 2-manifold is a locally connected metric space that is not separated by a finite set. This fact, with Theorem 1, shows that the conditions above are necessary.

Theorem 1. *Each 2-manifold (perhaps with boundary) is unifoliate.*

Proof. The property of being unifoliate is a local topological property, so consideration of the following two cases in a Cartesian plane is adequate: (1) P is the origin, and D is the interior of the circle $x^2 + y^2 = 1$; (2) P is the origin, and D is the set of all (x, y) such that $x^2 + y^2 < 1$ and $y \leq 0$. In each of these cases the proof can be carried through by use of known properties of the plane. An accessible reference is [2], Chapter IV, especially Theorem 106 and Theorem 113.

By the boundary of a 2-cell will be meant the simple closed curve contained in the 2-cell that does not separate the 2-cell. By an annulus will be meant a set that is the sum of two 2-cells whose common part is the sum of two arcs that is contained in the boundary of each 2-cell such that the closure of the sum of the boundaries of the 2-cells minus their common part is the sum of two mutually exclusive simple closed curves. These two simple closed curves will be called the boundary of the annulus. The definition of a Möbius strip is identical except that the set that is to be the boundary is in this case a simple closed curve. By a Klein bottle is meant a set that is the sum of two Möbius strips whose common part is the boundary of each Möbius strip.

Received March 20, 1956; in revised form, November 20, 1956.