THE THEOREMS OF LEDERMANN AND OSTROWSKI ON POSITIVE MATRICES

By Alfred Brauer

A square matrix $A = (a_{\kappa\lambda})$ of order n is called positive if all its elements are positive. 0. Perron [6] proved that the absolute greatest characteristic root of a positive matrix is positive and greater than the moduli of all the other roots. G. Frobenius [2] gave another proof of this theorem. Moreover, he obtained the following results. The absolute greatest root ω of a positive matrix is simple. The coordinates of a characteristic vector belonging to ω can be chosen all as positive numbers. Let $a = \max a_{\kappa\kappa}$ be the maximum of the elements of the main diagonal of A. Set

$$R_\kappa = \sum_{\nu=1}^{n} a_{\kappa\nu} \qquad (\nu = 1, 2, \cdots, n),$$

$$R = \max R_\kappa \quad \text{and} \quad r = \min R_\kappa .$$

Frobenius proved that ω satisfies the inequalities

$$R \geq \omega \geq r \tag{1}$$

and

$$\omega > a. \tag{2}$$

These inequalities follow at once from the fact that the system of linear equations

$$\omega y_\lambda = \sum_{\nu=1}^{n} a_{\nu\lambda} y_\nu \qquad (\lambda = 1, 2, \cdots, n) \tag{3}$$

has a positive solution. Adding the equations we obtain

$$\sum_{\lambda=1}^{n} \omega y_\lambda = \sum_{\lambda=1}^{n} \sum_{\nu=1}^{n} a_{\nu\lambda} y_\nu = \sum_{\nu=1}^{n} y_\nu \sum_{\lambda=1}^{n} a_{\nu\lambda} = \sum_{\nu=1}^{n} y_\nu R_\nu \leq R \sum_{\nu=1}^{n} y_\nu .$$

Dividing by $\sum_{\nu=1}^{n} y_\nu$ we obtain $\omega \leq R$. Similarly we can prove that $\omega \geq r$. Assume that $a = a_{kk}$. Writing the k-th of the equations (3) in the form

$$(\omega - a) y_k = (\omega - a_{kk}) y_k = \sum_{\substack{\nu=1 \\ \nu \neq k}}^{n} a_{\nu k} y_\nu \tag{4}$$

we obtain (2) since the right hand of (4) is positive. (See O. Taussky [7].)

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