DIFFERENTIABLE APPROXIMATIONS TO INTERIOR FUNCTIONS

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1. **Introduction.** This paper is a continuation of [2]. It is based on the main theorem of that paper. This theorem will be referred to in the present paper as Theorem A.

Theorem A. *Let D be a domain in the plane with closure D^- and let $f:D^- \to R^2$ be continuous on D^- and light interior on D into the plane. Let n be a positive integer. Then for every positive number ϵ there exists a continuous function $g:D^- \to R^2$ such that* (1) *g is light interior on D into R^2,* (2) *$f(z) = g(z)$ for z on the boundry of D,* (3) *g is n times continuously partially differentiable,* (4) *g has the same topological critical points as f, and* (5) *$|g(z) - f(z)| < \epsilon$ for all z in D^-.*

The pattern of the approximation theorems of this paper can be stated roughly as follows. Given a domain D in the plane and a function f defined on D (or on the closure of D) and being given certain topological properties for f, we show that f can be approximated uniformly by functions having the same topological properties as f and in addition having n continuous partial derivatives. We prove such theorems in this paper for compact 1-monotone mappings, ordinary monotone mappings, and real-valued interior functions.

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2. **Homeomorphisms and monotone mappings.** Before making use of Theorem A we must first make some remarks concerning Theorem A and its proof. The proof of this theorem actually proved somewhat more than was stated. It was proved that the approximating function g was not only closer to f than some preassigned number ϵ but also that $|f(z) - g(z)|$ was less than a certain continuous real-valued function $\epsilon(z)$ vanishing exactly on the boundary of the domain D and having ϵ as a maximum. The only use made of the values of f on the boundary was to define g to be equal to f there. Thus it follows from continuity that if f were to be defined and continuous not on the whole of D^- but only on $D \cup K$ where K is a subset (possibly empty) of the boundary of D, then the approximating function g would satisfy all the conclusions of Theorem A with the set D^- replaced by the set $D \cup K$. In particular, if conclusion (2) be deleted, the boundary need not be mentioned at all. This remark applies for the same reason to Theorems 1 and 5 of the present paper. Further, an examination of the proof given for Theorem A shows that the image $f(D)$ was

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