MASS DISTRIBUTIONS FOR PRODUCTS OF SUBHARMONIC FUNCTIONS

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1. **Introduction.** To obtain a representation for the mass distributions for logarithmic-subharmonic functions, or functions of class PL, M. O. Reade [11] and O. Ishikawa [7] have derived certain properties of mass distributions for products of subharmonic functions. One of their results states that if u_1, u_2, and $u_1 u_2$ are continuous subharmonic functions on a plane region Ω, then for all bounded Borel sets E having closure in Ω

$$(1.1) \qquad m(E) = \int_E u_1\, dm_2 + \int_E u_2\, dm_1 - \frac{1}{\pi} \int_E \left(\frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial y} \right) da,$$

where m_1, m_2, and m are the respective mass distributions and a denotes Lebesgue plane measure.

A relationship of this sort is to be expected from the outset, in view of its evident validity for functions in C''. However, Reade and Ishikawa leave open the question as to its validity for the case of discontinuous subharmonic functions. (Ishikawa does not mention continuity, but the proofs indicate that his definition of subharmonicity is one requiring subharmonic functions to be continuous.)

Moreover, the hypothesis of subharmonicity does not occur in the C'' case, and this fact leads us to seek some further weakening of the conditions on u_1, u_2, and $u_1 u_2$ in the general case. A minimum requirement on u_1, u_2, and $u_1 u_2$ would seem to be that they be δ-subharmonic functions, i.e. functions representable as differences of subharmonic functions (see [1]), or at least almost δ-subharmonic functions.

In this connection a natural class of functions to consider is the family $\mathfrak{A}$ of all functions representable as differences of locally bounded subharmonic functions on Ω, since this class is closed under multiplication and thus forms an algebra [1; 347]. Drawing on the recent work of Brelot [3] and Deny [4], we establish (1.1) for arbitrary functions u_1, u_2 in $\mathfrak{A}$. The technique employed is to introduce a quadratic functional (analogous to the Dirichlet integral) in terms of the energy integral and the product mass, and then to prove that the functional defined in this way actually coincides with the Dirichlet integral.

It follows from Theorem 28 of [1; 351] that if w is a function in $\mathfrak{A}$ and φ is a real-valued function having a Lipschitzian derivative on an interval containing the range of w, then the composite function $\varphi \circ w$ is almost δ-subharmonic on

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