# A GENERALIZATION OF COMPLETELY CONVEX FUNCTIONS

By M. H. Protter

1. **Introduction.** A function $f(x)$ defined in $(a, b)$ is *completely convex* if it possesses derivatives of all orders and

$$f^{(4k)}(x) \geq 0, \qquad f^{(4k+2)}(x) \leq 0, \qquad a \leq x \leq b, \qquad k = 0,1,2, \cdots \tag{1}$$

This term was introduced by Widder who proved the following result [4].

THEOREM (Widder). *If $f(x)$ is completely convex in an interval $(a, b)$, it is an entire function of exponential type.*

This result was extended considerably by Boas and Pólya [2] who examined the analytic character of functions as influenced by an alternation in signs of sequences of derivatives.

It is the purpose of this paper to extend Widder's theorem in a somewhat different direction. A function $f(x)$ defined in an interval $(a, b)$ is ***almost completely convex*** if it possesses derivatives of all orders and

$$\begin{gathered} f^{(4k)}(x) \geq 0, \qquad a \leq x \leq b \\ f^{(4k+2)}(a) + f^{(4k+2)}(b) \leq \frac{\pi^2}{(b-a)^2}\,[f^{(4k)}(a) + f^{(4k)}(b)], \quad k = 0,1,2, \cdots . \end{gathered} \tag{2}$$

Examples of completely convex functions are $\sin x$, $0 \leq x \leq \pi$, and $\cos x$, $-\pi/2 \leq x \leq \pi/2$. The extensions of Boas and Pólya relax condition (1) so that a sequence $\{n_k\}$ of derivatives is non-negative while an intermediate sequence of the form $\{n_k + 2q_k\}$ of derivatives is non-positive. On the other hand, condition (2) above assumes that the derivatives of order $4k + 2$ at the end points satisfy certain growth conditions. It may happen that all derivatives of a function are non-negative throughout $(a, b)$, and the function still is almost completely convex. For example $e^{cx}$, $0 \leq x \leq 1$, $0 < c \leq \pi$, is such a function.

In §2 the following theorem is established.

THEOREM 1. *If $f(x)$ is almost completely convex in an interval $(a, b)$, it is an entire function of exponential type at most $\pi/(b - a)$.*

An elementary proof of Widder's theorem was given by Boas [1]. The proof of Theorem 1 is obtained by an extension of the technique employed in Boas' proof.

In §3 it is shown how the notion of completely convex function can be extended to the case of more than one independent variable.

Received July 14, 1956. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF 18(600)-1117.