ALGEBRAIC EXTENSIONS OF ARBITRARY FIELDS

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The preceding paper [5] suggests two questions. First, is the condition that a field has no extension of degree divisible by p equivalent to the condition that it has no extension of degree p? Second, if a field has a cyclic extension of degree p does it have a cyclic extension of degree p^ν for every ν? These questions are answered by Theorems 1 and 2 below. It turns out that they can be answered by Galois theory only, without any assumption about the nature of the ground field.

Theorem 1. *Let n be any positive integer. There exists a field K which has algebraic extensions of degree divisible by n but has no extension of degree $\leq n$.*

Proof. Let n be given. Choose m such that $m \geq 5$, n divides $m!/2$, and $n! < m!/2$. Let k be any field which has a normal separable extension E/k with the alternating group on m letters, $\mathfrak{A}$, as its Galois group. I shall show that k has an algebraic extension K with the desired property.

Let k and E be given. E, and all fields mentioned in the rest of this proof shall be understood to be subfields of some fixed algebraic closure of k. Consider the set of all extensions L of k such that

$$E \cap L = k. \tag{1}$$

This is equivalent to the condition that EL/L has the same Galois group $\mathfrak{A}$ as E/k. This set is partially ordered under inclusion. It is not empty because it at least contains k itself. From (1) it is evident that the union of any linearly ordered subset is again in the set. So it contains a maximal element K, and the Galois group of EK/K is $\mathfrak{A}$. Let N be any normal extension of K. Then since K was maximal, the Galois group of EN/N is some proper subgroup of $\mathfrak{A}$; since $EK \cap N$ is normal over K, it is an invariant subgroup; since $\mathfrak{A}$ is simple, it is the subgroup $\{1\}$. Thus every normal extension of K contains EK, so its degree over K is at least $m!/2 > n!$. Since any extension of degree $\leq n$ is contained in a normal extension of degree $\leq n!$, K has no such extension.

Remark. K is very far from being algebraically closed. For example, if p is any odd prime dividing m, then E has a subfield over which E is cyclic of degree p. By Theorem 2, this subfield has a cyclic extension of degree p^ν for every ν. Since $\mathfrak{A}$ has an element of period 4, this is also true for $p = 2$. So K has extensions of degree $2^\lambda 3^\mu 5^\nu m!$ for every $\lambda \geq -1$, $\mu \geq 0$, $\nu \geq 0$.

The second question may be sharpened by asking: When does a field k have a cyclic extension of degree p^∞? An (infinite) algebraic extension C_∞/k is called *cyclic of degree p^∞* if C_∞ is the union of a chain of subfields C_ν with C_ν/k cyclic

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