COMPLETE SEQUENCES AND APPROXIMATIONS IN NORMED LINEAR SPACES

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1. **Introduction.** Let X denote a (real or complex) normed linear space, and let X^* denote its conjugate space. As usual, a sequence $\{f_n\}$ of elements in X is said to be *complete* (also "closed" or "total" in the literature), if $\varphi = 0$ is the only $\varphi \in X^*$ satisfying $\varphi(f_n) = 0$ for every n. This paper deals with the following types of completeness.

DEFINITION 1. Given a sequence $\{a_n\}$ of non-negative numbers, a sequence $\{f_n\}$ of elements of X is said to be *$\{a_n\}$-complete*, if $\varphi = 0$ is the only $\varphi \in X^*$ satisfying $| \varphi(f_n) | \leq a_n$ $(n = 1, 2, 3, \cdots)$.

DEFINITION 2. Let $p \geq 1$. A sequence $\{f_n\}$ in X is said to be *complete of order* p, if for $\varphi \in X^*$, the convergence of the series $\sum_{n=1}^{\infty} | \varphi(f_n) |^p$ implies $\varphi = 0$. In particular, $\{f_n\}$ is *complete of order* ∞, if $\varphi = 0$ is the only $\varphi \in X^*$ for which the sequence $\{\varphi(f_n)\}$ is bounded.

Clearly each of these types of completeness implies the usual completeness. If we denote by $\{0\}$ the sequence formed by zeros only, then the usual completeness is precisely the $\{0\}$-completeness. If $\{1\}$ denotes the sequence with all terms unity, then the completeness of order ∞ is equivalent to the $\{1\}$-completeness. It is also clear that, if $\infty \geq p_1 \geq p_2 \geq 1$, then completeness of order p_1 implies completeness of order p_2. When $a_n > 0$ for every n, the $\{a_n\}$-completeness of $\{f_n\}$ obviously coincides with the $\{1\}$-completeness of $\{f_n/a_n\}$.

The new types of completeness will be characterized by certain approximation properties in §2. In §3 we shall give a theorem for constructing $\{a_n\}$-complete sequences. Then examples in function spaces will be given in §4. In the final §5, we shall prove theorems of Paley-Wiener type [9; 100-108] for $\{a_n\}$-completeness and completeness of order p.

2. **Approximation theorems.** We first study approximation properties related to the new types of completeness.

THEOREM 1. *Let $\{a_n\}$ be a sequence of non-negative numbers. A sequence $\{f_n\}$ of elements in a normed linear space X is $\{a_n\}$-complete, if and only if, for any*

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