UNIQUENESS OF MAPPING PAIRS FOR ELLIPTIC EQUATIONS

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1. **Introduction.** Let D^* and Δ^* be simple closed plane Jordan curves with interiors D and Δ in the (x, y)- and (X, Y)-planes. Let $z^{(1)}, z^{(2)}, z^{(3)}$ be distinct points on D^*; and let $Z^{(1)}, Z^{(2)}, Z^{(3)}$ be distinct points on Δ^* in the same order on Δ^* as the points $z^{(1)}, z^{(2)}, z^{(3)}$ on D^*.

A pair of functions $u(x, y)$, $v(x, y)$ which (a) give a 1:1 and continuous mapping, $X = u(x, y)$, $Y = v(x, y)$, of $D \cup D^*$ onto $\Delta \cup \Delta^*$ with $z^{(j)}$ going into $Z^{(j)}$, $j = 1, 2, 3$, and (b) satisfy in D a prescribed elliptic system,

$$\alpha u_x + \beta u_y = v_y , \qquad \gamma u_x + \delta u_y = - v_x , \tag{1.1}$$

$$0 < \alpha, \qquad 0 < \alpha\delta - \tfrac{1}{4}(\beta + \gamma)^2, \tag{1.2}$$

is known as a mapping pair for the system. If u, v gives a mapping of the type (a), and if (c) the first partial derivatives of u and v exist almost everywhere in D and are locally square integrable, and if u and v satisfy (1.1) almost everywhere in D, then u, v is a generalized mapping pair. The problem of determining a mapping pair, or generalized mapping pairs, for a prescribed elliptic system is an extension of the Riemann problem.

The existence of a generalized mapping pair has been proved with light restrictions on the coefficients α, β, γ, δ of (1.1) and without additional restrictions on D^* and Δ^*. In a recent paper [3] Bers and Nirenberg pointed out that if α, β, γ, δ are functions of x, y alone, then a generalized mapping pair for the system (1.1), (1.2) exists if α, β, γ, δ are bounded and measurable on D. They also prove the existence of a generalized mapping pair if α, β, γ, δ are continuous and bounded functions of x, y, u, v for $(x, y) \in D$ and $(u, v) \in \Delta$.

The uniqueness phase of the mapping problem is not in such an advanced stage. For the special case α, β, γ, δ functions of x, y alone and $\beta = \gamma$, $\alpha\delta - \beta^2 = 1$, Morrey [12] has given both uniqueness and existence theorems under the assumptions that α, β, γ, δ are bounded and measurable on D. Under the less restrictive conditions (1.2), all uniqueness theorems to date have imposed strong continuity restrictions on both the bounding curves of the domains being mapped and the coefficients of the differential equations which the mapping pair u, v must satisfy. For example, Lavrent'ev [10] states a uniqueness result under the assumptions that $\alpha = \alpha\ (x, y; u, v), \cdots , \delta = \delta\ (x, y; u, v)$ possess uniformly continuous second partial derivatives on D and Δ, and the boundary curves D^* and Δ^* are "smooth" curves. Gergen and Dressel [7], [8] have proved

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