LOCAL C^n TRANSFORMATIONS OF THE REAL LINE

BY SHLOMO STERNBERG

1. Let T^n be the class of all monotone increasing functions f of a real variable, defined in some neighborhood of the origin, satisfying $f(0) = 0$, and of class C^n. We shall regard such functions as local transformations of the real line into itself. We can compose two functions of the same class to obtain a third function of that class provided we restrict attention to a sufficiently small neighborhood of the region. That is, if $f, g \varepsilon T^n$ then $f(g(x))$ is defined for all those x in the domain of g such that $g(x)$ is in the domain of f. In order to avoid difficulties about the domain, we may introduce the equivalence relation "$f \equiv g$ on some neighborhood of the origin." Since composition of two functions is an invariant operation with respect to this equivalence relation, this then defines a binary operation on the coset space $\mathbf{T}^n$. If, for $n \geq 1$ we restrict ourselves to those functions satisfying $f'(0) \neq 0$, we have a group $\mathbf{G}^n$.

If $y = g(x)$ is an element of T^n (and $g'(0) \neq 0$), we can consider it as a change of coordinates. Every function $f \varepsilon T^n$ (with suitable restrictions of domain) when expressed in the new coordinates has the form $g[f(g^{-1}[\cdot])]$. Thus a change of coordinates $y = g(x)$ corresponds to an inner automorphism of the group $\mathbf{G}^n$ by $\mathbf{g}$ where $\mathbf{g}$ is the coset containing g. The purpose of this article is to study the invariants of $\mathbf{G}^n$ with respect to inner automorphisms, i.e. those properties of a function which are purely local and independent of the coordinate system.

2. The problem will be split into two parts: $n = 0$ and $n > 0$. In the former (and simpler) case, a complete set of invariants will be contained for all conjugacy classes of G^0. For $n > 0$ we shall only deal with those classes satisfying $f'(0) \neq 1$, where f is an element of a coset of the conjugacy class. (The derivative at the origin is clearly a local property and is, furthermore, independent of the choice of coordinates.) In both cases the underlying idea is to write the functional equation $g^{-1}fg = h$ as $f[g(x)] = g[h(x)]$. The latter equation implies that the value of g at the point x determines its value at the point $h(x)$.

3. In the continuous case, it is clear that the behaviour of a function to the right of the origin is completely independent of its behaviour to the left. We may thus restrict attention to the right of the origin in obtaining invariants, adding a similar set for the left to obtain a complete set of invariants. We thus consider the class of continuous monotone increasing functions defined on some interval $(0, b)$ and satisfying $f(0) = 0$.

Suppose that f is transformed into h by a change of coordinates g, i.e. $gfg^{-1} = h$. It is clear that the closed set A_f of invariant points of the function f is mapped by g into the set A_h of invariant points for h. If we write the functional equation

Received November 9, 1955; in revised form, September 6, 1956.