WEAK BOUNDARY COMPONENTS OF AN OPEN RIEMANN SURFACE

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Introduction. Ahlfors and Beurling [1] have shown that the existence of univalent functions with a finite Dirichlet integral on a planar Riemann surface R is equivalent to the existence of bounded univalent functions on R. Moreover, Sario [6] has established that there are no such functions on R if and only if every ideal boundary component is weak, i.e., has vanishing capacity. It is of interest therefore to find tests for the weakness of a boundary component. This paper is concerned with finding such tests, and the criteria obtained are valid not only for planar surfaces but for arbitrary open Riemann surfaces as well.

§1 contains the reduction thereom which enables us to solve certain extremal problems on the entire surface, once we know their solution for compact subregions. In §2 the concept of an ideal boundary component is defined, and §3 contains the fundamental existence theorem for the capacity function. The notion of weak boundary component is defined, and its relation to the behavior of univalent functions is discussed in §4.

In §5 the modulus of a ring domain is defined and a modular criterion proved. Regular chains are considered in §6, and a test for a weak component is expressed in terms of them. §7 deals with a conformal metric and a related weakness criterion.

In §8 covering surfaces of the complex plane which are ramified over a finite number of points are considered. A theorem is proved which shows that the weakness of a boundary component is related to the ramification of the surface. §9 contains two weakness criteria for plane point sets. One criterion uses the notion of relative width, the other makes use of the method of square nets.

1. **The reduction theorem.** For completeness we state the following theorem whose proof is found in [5]. Let R be an open Riemann surface and C a class of functions defined on R. Let $\{R_n\}$ be an exhaustion of R. On R_n, let C_n be a class of functions such that the restriction of $p \in C_n$ to R_{n-1} belongs to the class C_{n-1}. It is assumed that this relation also holds for R, R_n and C, C_n.

For R_n and $p \in C_n$, a functional $m(R_n, p)$ is given, such that, for $p' \in C_n$ tending uniformly on R_n to $p'' \in C_n$, $m(R_n, p')$ tends to $m(R_n, p'')$. We define the functional $m(R, p)$ for R and $p \in C$ by

$$m(R, p) = \lim_{n\to\infty} m(R_n, p).$$

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